Topology Qualifying Exam

January, 2014

Justify all the calculations and state the theorems you use in your answers.

1. (10 pts) Prove that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

2. (10 pts) Let X be a topological space, and let $\Delta = \{(x, y) \in X \times X \mid x = y\}$. Show that X is Hausdorff if and only if Δ is closed in $X \times X$.

3. (10 pts) Suppose that $X \subseteq Y$ and X is a deformation retract of Y. Show that if X is a path connected space, then Y is path connected.

4. a) (5 pts) Give the definition of a covering space $\hat{X}$ (and covering map $p : \hat{X} \to X$) for a topological space X.

   b) (15 pts) State the homotopy lifting property of covering spaces. Use it to show that a covering map induces an injection on fundamental groups.

5. (10 pts) Find all surfaces, orientable and non-orientable, which can be covered by a closed surface (i.e. compact with empty boundary) of genus 2. Prove that your answer is correct.

6. Let X be a space obtained by attaching two 2-cells to the torus $S^1 \times S^1$, one along a simple closed curve $\{x\} \times S^1$ and the other along $\{y\} \times S^1$ for two points $x \neq y$ in S^1 .

   a) (10 pts) Draw an embedding of X in $\mathbb{R}^3$ and calculate its fundamental group.

   b) (10 pts) Calculate homology groups of X.

7. (10 pts) Use cellular homology to calculate the homology groups of $S^n \times S^m$.

8. (10 pts) Show that a map $S^n \to S^n$ has a fixed point unless its degree is equal to the degree of the antipodal map $a : x \mapsto -x$.