Topology Qualifying Exam, Spring 2021

Please attempt **8 out of 9** problems and clearly mark the one not to be graded.

1. Let X be a topological space, give $X \times X$ the product topology, and let

$$\Delta := \{(x, x) \in X \times X \mid x \in X\}.$$

Show that the following are equivalent:

(i) The set Δ is closed in $X \times X$.

(ii) The space X is Hausdorff.

2. Let $X = \prod_{n=1}^{\infty} \{0, 1\}$, endowed with the product topology.

(a) Show that for all points $x, y \in X$ with $x \neq y$, there are open subsets U_x, U_y of X such that $x \in U_x$, $y \in U_y$, $U_x \cup U_y = X$, and $U_x \subset U_y = \varnothing$.

(b) Show that X is totally disconnected: the only nonempty connected subsets of X are $\{x\}$ for $x \in X$.

3. For nonempty subsets A and B of a metric space (X, d) the **setwise distance** is $d(A, B) := \inf\{d(a, b) | a \in A,\ b \in B\}$.

(a) Suppose that A and B are compact. Show that there is $a \in A$ and $b \in B$ such that $d(a, b) = d(A, B)$.[1]

(b) Suppose that A is closed and B is compact. Show: if $d(A, B) = 0$ then $A \cap B \neq \varnothing$.

(c) Give an example in which A is closed, B is compact, and $d(a, b) > d(A, B)$ for all $a \in A$ and $b \in B$. (Suggestion: take $X = \{0\} \cup (1, 2] \subset \mathbb{R}$.)

4. Suppose that X is a topological space, that $x_0 \in X$, and that every continuous map $\gamma \colon S^1 \to X$ is freely[2] homotopic to the constant map to x_0. Prove that $\pi_1(X, x_0) = \{e\}$.

5. Identify five mutually non-homeomorphic connected spaces X for which there is a covering map $p : X \to K$, where K is the Klein bottle, giving an example of a corresponding covering in each case.

6. For each of the following spaces, compute the fundamental group and the homology groups.

(a) The graph Θ consisting of two vertices and three edges connecting these vertices.

(b) The two-dimensional cell complex Θ_2 consisting of a closed circle and three two-dimensional disks each having boundary running once around that circle.

7. Prove directly from the definition that the zeroth singular homology of a nonempty path-connected space is isomorphic to $\mathbb{Z}$.

8. Let $\Sigma_{g,n}$ denote the compact oriented surface of genus g with n boundary components.

(a) Show that $\Sigma_{0,3}$ and $\Sigma_{1,1}$ are both homotopy equivalent to $S^1 \vee S^1$.

(b) Give a complete classification of pairs (g, n) and (g', n') for which $\Sigma_{g,n}$ is homotopy equivalent to $\Sigma_{g',n'}$.

9. Prove that for every continuous map $f \colon S^{2n} \to S^{2n}$ there is $x \in S^{2n}$ so that either $f(x) = x$ or $f(x) = -x$. You may use standard facts about degrees of maps of the sphere, including that the antipodal map of S^{2n} has degree -1.

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[1] Throughout this problem you may use without proof the continuity of $d : X \times X \to \mathbb{R}$.

[2] i.e., with no conditions on basepoints.