Topology Qualifying Exam
August, 2018

(1) Prove that a finite CW complex must be Hausdorff.

(2) Let X be a compact space and let $f : X \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x, 0) > 0$ for all $x \in X$. Prove that there is $\epsilon > 0$ such that $f(x, t) > 0$ whenever $|t| < \epsilon$. Moreover give an example showing that this conclusion may not hold if X is not assumed compact.

(3) Let

$$X = \{(x, y) \in \mathbb{R}^2 | x > 0,\ y \geq 0,\ \text{ and } \frac{y}{x} \text{ is rational}\},$$

and equip X with the subspace topology induced by the usual topology on $\mathbb{R}^2$. Prove or disprove that X is connected.

(4) Prove the following portion of van Kampen's Theorem: If $X = A \cup B$ and A, B, and $A \cap B$ are open and path connected with $* \in A \cap B$, then there is a surjection $\pi_1(A, *) * \pi_1(B, *) \to \pi_1(X, *)$.

(5) For each $n \in \mathbb{Z}$, give an example of a map $f_n : S^2 \to S^2$ of degree n. For which n must your example have a fixed point?

(6) Let C be a cylinder. Let I and J be disjoint closed intervals that are contained in ∂C. What is the Euler characteristic of the surface S obtained by identifying I and J? Can all surfaces with non-empty boundary and with this Euler characteristic be obtained from this construction?

(7) Recall that a covering space $\pi : \tilde{X} \to X$ between path-connected spaces is called *normal* (or *regular*) provided that, for one and hence any $x_0 \in X$, the deck transformation group of π acts transitively on $\pi^{-1}(x_0)$. Give examples both of a 3-sheeted normal covering space of $S^1 \vee S^1$, and of a path-connected 3-sheeted non-normal covering space of $S^1 \vee S^1$.

(8) Compute the homology of the subset $X \subset \mathbb{R}^3$ formed as the union of the unit sphere, the z-axis, and the xy-plane.