Topology Qualifying Exam Aug 2020

(1) Prove that a compact subset of a Hausdorff space is closed, and give a counterexample showing that the Hausdorff assumption is necessary.

(2) Recall that the **one-point compactification** $\bar{X}$ of a topological space X is given by the following.

- As a set $\bar{X} = X \bigsqcup \{\infty\}$, where ∞ is a point.
- A subset is open in $\bar{X}$ if it is either an open set of X, or is of the form $U \bigsqcup \{\infty\}$ where $U \subset X$ has the property that $X - U$ is compact.

Prove that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the sphere S^n.

(3) Consider two 2-spheres Σ_1 and Σ_2 with their respective north and south poles N_1, N_2 and S_1, S_2. Take the quotient space $A := \Sigma_1 \sqcup \Sigma_2 / \sim$, where $N_1 \sim N_2$ and $S_1 \sim S_2$. Describe all covering spaces of A.

(4) Show that the torus is a double cover of the Klein bottle. Show also that the double covering map is not homotopic to the constant map.

(5)
- Show that there is no continuous map $f : S^2 \to S^2$ such that $f \circ f$ has degree -1.
- Find a continuous map $f : T^2 \to T^2$ so that the induced map $(f \circ f)_* : H_1(T^2) \to H_1(T^2)$ is given by multiplication by -1.

(6) Let $A \subset X$ be a pair of topological spaces. Recall that a map $r : X \to A$ is called a *retraction* if $r(a) = a$ for all $a \in A$. Assume such a retraction exists below.

- If $X = D^2$, show that $H_i(A) = 0$ for $i > 0$ and $H_0(A) = \mathbb{Z}$.
- If X is an annulus, list all possible $H_i(A)$ for $i \geq 0$. Prove that your list is exhaustive.

(7) For a topological space X, the **suspension** $S(X)$ is defined as $([0,1] \times X)/\sim$, where $(c, x) \sim (c, x)$ for either $c = 0$ or $c = 1$.

- Prove that $S(S^n)$ is homeomorphic to S^{n+1}.
- Find $H_1(S(T^2))$.

(8) For a map $f : X \to Y$, define the **graph of** f as

$$G(f) := \{(x, y) \in X \times Y : y = f(x)\}.$$

Let $f : S^1 \to S^1$ be a continuous map with degree d. Find, in terms of d, the relative homology $H_*(S^1 \times S^1, G(f))$.