Real analysis qualifying exam, Fall 2021

(1) Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $x_1 > 0$ and
$$x_{n+1} = 1 - (2 + x_n)^{-1} = \frac{1 + x_n}{2 + x_n}.$$
Prove that the sequence $\{x_n\}$ converges, and find its limit.

(2) a) Let $F \subset \mathbb{R}$ be closed, and define
$$\delta_F(y) := \inf_{x \in F} |x - y|.$$
For $y \notin F$, show that
$$\int_F |x - y|^{-2}\, dx \le \frac{2}{\delta_F(y)}.$$

b) Let $F \subset \mathbb{R}$ be a closed set whose complement has finite measure, i.e. $m(\mathbb{R} \backslash F) < \infty$. Define the function
$$I(x) := \int_{\mathbb{R}} \frac{\delta_F(y)}{|x - y|^2}\, dy.$$

Prove that $I(x) = \infty$ if $x \notin F$, however $I(x) < \infty$ for almost every $x \in F$. (Hint: investigate $\int_F I(x)\, dx$.)

(3) Recall that a set $E \subset \mathbb{R}^d$ is measurable if for every $\epsilon > 0$ there is an open set $U \subset \mathbb{R}^d$ such that $m^*(U \backslash E) < \epsilon$.

(a) Prove that if E is measurable then for all $\epsilon > 0$ there exists an elementary set F, such that $m(E \Delta F) < \epsilon$. Here $m(E)$ denotes the Lebesgue measure of E, a set F is called elementary if it is a finite union of rectangles and $E \Delta F$ denotes the symmetric difference of the sets E and F.

(b) Let $E \subset \mathbb{R}$ be a measurable set, such that $0 < m(E) < \infty$.
Use part (a) to show that,
$$\lim_{n \to \infty} \int_E \sin(nt)\, dt = 0.$$

(4) Let f be a measurable function on $\mathbb{R}$. Show that the graph of f has measure zero in $\mathbb{R}^2$.

(5) Consider the Hilbert space $\mathcal{H} = L^2([0, 1])$.

(a) Prove that of $E \subset \mathcal{H}$ is closed and convex then E contains an element of smallest norm. *Hint:* Show that if $\|f_n\|_2 \to \ min\{f \in E : \ \|f\|_2\}$ then $\{f_n\}$ is a Cauchy sequence.

(b) Construct a non-empty closed subset $E \subset \mathcal{H}$ which does not contain an element of smallest norm.