Real Analysis Qualifying Examination

Spring 2022

The six problems on this exam have equal weighting. To receive full credit give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

Notation: In Questions 3 and 6 below m denotes Lebesgue measure on $\mathbb{R}$.

1. Prove that
$$f(x) = \sum_{n=1}^{\infty} \frac{x}{1+n^2x^2}$$
defines a function that is differentiable with a continuous derivative on $(0, \infty)$ and that
$$f'(x) = \sum_{n=1}^{\infty} \frac{1-n^2x^2}{(1+n^2x^2)^2}$$
on $(0, \infty)$.

2. (a) Let E be a subset of $\mathbb{R}^d$ with the property that $E \cap \{x \in \mathbb{R}^d : |x| \leq k\}$ is closed for all $k \in \mathbb{N}$. Prove that E must itself be closed.
   (b) Let μ be a Borel measure on $\mathbb{R}^d$ that assigns finite measure to all bounded Borel sets. Prove that for every F_σ set $V \subseteq \mathbb{R}^d$ and $\varepsilon > 0$ there exists a closed set $F \subseteq V$ such that $\mu\big(V \setminus F\big) < \varepsilon$.

3. Let $f \in L^1(\mathbb{R})$. Prove that
$$\lim_{n\to\infty} \int_{\mathbb{R}} |f(x)|^{1/n}\, dx = m(\{x \in \mathbb{R} : f(x) \neq 0\})$$
where we are allowing for the possibility that both sides equal infinity.

4. Let $f, g \in L^2([0,1])$. Prove that if
$$\int_0^1 f(x)\, x^n\, dx = \int_0^1 g(x)\, x^n\, dx$$
for all integers $n \geq 0$, then $f = g$ almost everywhere.

5. Prove that if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, then
$$f * g(x) := \int_{\mathbb{R}} f(x-y)g(y)\, dy$$
defines a function in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ that satisfies the following estimates:
   (a) $\|f * g\|_1 \leq \|g\|_1 \|f\|_1$
   (b) $\|f * g\|_2 \leq \|g\|_1 \|f\|_2$

   Hint: For the second estimate first argue that $|f * g|^2 \leq \|g\|_1 (|f|^2 * |g|)$

6. (a) Prove that if $E \subseteq \mathbb{R}$ with $m(E) > 0$, then
$$\int_E e^{-\pi x^2}\, dx > 0.$$
   (b) Let $f \in L^\infty(\mathbb{R})$. Prove that
$$\lim_{p\to\infty} \left(\int_{\mathbb{R}} |f(x)|^p\, e^{-\pi x^2}\, dx \right)^{1/p} = \|f\|_\infty.$$