By providing my signature below, I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else:

Name (sign): ____________________ Name (print): Solutions

Student Number: ____________________

Instructor's Name: ____________________ Class Time: ____________________

Problem Number	Points Possible	Points Earned
1	22	
2	15	
3	18	
4	8	
5	15	
6	18	
7	14	
8	10	
9	25	
10	14	
11	10	
12	10	
13	15	
14	16	
15	20	
Total:	230	

- If you need extra space use the last page. *Do not tear off the last page!*
- Please show your work. **An unjustified answer may receive little or no credit.**
- If you make use of a theorem to justify a conclusion, then state the theorem used by name.
- Your work must be **neat**. If I can't read it (or can't find it), I can't grade it.
- The total number of possible points that is assigned for each problem is shown here. The number of points for each subproblem is shown within the exam.
- You may not use a cell phone or smart watch. Please turn off your cell phones, and store them and your smart watches away from your desk or table.
- You are only allowed to use a **TI-30XS Multiview** calculator; the name must match exactly. No other calculators are permitted, and sharing of calculators is not permitted.
- You do not have to use a calculator; answers containing symbolic expressions such as $\cos(\pi/3)$ and $\ln(e^4)$ are acceptable. Include an exact answer for each problem.

1. Determine the following limits; briefly explain your thinking on each one. If you apply L'Hopital's rule, indicate where you have applied it and why you can apply it. If your final answer is "does not exist," ∞, or $-\infty$, briefly explain your answer. (You will not receive full credit for a "does not exist" answer if the answer is ∞ or $-\infty$.)

(a) [4 pts] $\lim_{x\to 1} \frac{x^2-9}{x^2+x-6}$

$\frac{1-9}{1+1-6} = \boxed{\frac{-8}{-4}}$ OR 2

I used direct sub (plugged in).

(b) [6 pts] $\lim_{x\to 2^+} \frac{x^2-9}{x^2+x-6} = \boxed{-\infty}$

$\frac{4-9}{4+2-6} = \frac{-5}{0}$ so limit DNE, may be $\pm\infty$

$x^2-9 \to -5$ as $x \to 2^+$
$x^2+x-6 \to 0$ and is positive as $x \to 2^+$

$\frac{-}{+}$ so $-\infty$

(c) [6 pts] $\lim_{p\to 0} \frac{\ln(1+5p)-p}{\sin(3p)} \overset{LH}{=} \lim_{p\to 0} \frac{\frac{5}{1+5p}-1}{3\cos(3p)} = \boxed{\frac{4}{3}}$

$\frac{\ln(1)-0}{\sin(0)} = \frac{0}{0}$ indeterminate form; can use L'Hopital's Rule (LH)

(d) [6 pts] $\lim_{x\to\infty} xe^{-2x} = \lim_{x\to\infty} \frac{x}{e^{2x}} \overset{LH}{=} \lim_{x\to\infty} \frac{1}{2e^{2x}} = \boxed{0}$

$\infty \cdot 0$ $\quad \frac{\infty}{\infty}$

$\infty \cdot 0$ is an indeterminate form; can't use LH yet
$\frac{\infty}{\infty}$ is an indeterminate form; can use LH

2. (a) [5 pts] State the limit definition of the derivative of $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(b) [10 pts] Use the **limit definition of the derivative** to determine the derivative of $f(x) = \dfrac{1}{3-2x}$. No points will be awarded for the application of differentiation rules (and L'Hopital's rule is not allowed).

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{3-2(x+h)} - \frac{1}{3-2x}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{3-2x-2h} - \frac{1}{3-2x}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{3-2x-(3-2x-2h)}{(3-2x-2h)(3-2x)}\right)$$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}}\left(\frac{2\cancel{h}}{(3-2x-2h)(3-2x)}\right)$$

$$= \frac{2}{(3-2x)^2}$$

— OR —

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{1}{x-a}\left(\frac{1}{3-2x} - \frac{1}{3-2a}\right)$$

$$= \lim_{x \to a} \frac{1}{x-a}\left(\frac{3-2a-(3-2x)}{(3-2x)(3-2a)}\right)$$

$$= \lim_{x \to a} \frac{1}{x-a}\left(\frac{2x-2a}{(3-2x)(3-2a)}\right)$$

$$= \lim_{x \to a} \frac{2\cancel{(x-a)}}{\cancel{(x-a)}(3-2x)(3-2a)}$$

$$= \frac{2}{(3-2a)^2}$$

3. Determine the first derivative of each of the following functions. Remember to use correct notation to write your final answer.

   (a) [6 pts] $f(x) = x^2 - \sin(x)\cos(x)$

$$f'(x) = 2x - \left(\sin(x)(-\sin(x)) + \cos(x)\cos(x)\right)$$

ok final answer

$$= 2x + \sin^2(x) - \cos^2(x)$$

   (b) [6 pts] $g(t) = \arctan(\ln(t))$

$$g'(t) = \frac{1}{1+(\ln(t))^2} \cdot \frac{1}{t}$$

ok final answer

$$= \frac{1}{t + t(\ln(t))^2}$$

   (c) [6 pts] $h(x) = \dfrac{e^{3x}}{x+1} = e^{3x}(x+1)^{-1}$

ok final answers

$$h'(x) = \frac{(x+1)(3e^{3x}) - e^{3x}(1)}{(x+1)^2}$$

OR

$$h'(x) = e^{3x}\left(-(x+1)^{-2}\right) + (x+1)^{-1}(3e^{3x})$$

$$= e^{3x}\left(-(x+1)^{-2} + (x+1)^{-1}(3)\right)$$

$$= (x+1)^{-2}(e^{3x})(-1 + 3(x+1))$$

$$= (x+1)^{-2}(e^{3x})(3x+2)$$

$$= \frac{3xe^{3x} + 3e^{3x} - e^{3x}}{(x+1)^2}$$

$$= \frac{3xe^{3x} + 2e^{3x}}{(x+1)^2}$$

4. [8 pts] Determine the **second derivative** of the function

$$f(t) = 3t^2 - \sqrt{t}$$

$$f'(t) = 6t - \tfrac{1}{2}t^{-1/2}$$

$$f''(t) = 6 + \tfrac{1}{4}t^{-3/2}$$

5. Consider the curve defined by the equation $3x^2 + 3xy + y^2 = 5$.

(a) [10 pts] Determine $\dfrac{dy}{dx}$.

$6x + 3x\frac{dy}{dx} + y(3) + 2y\frac{dy}{dx} = 0$

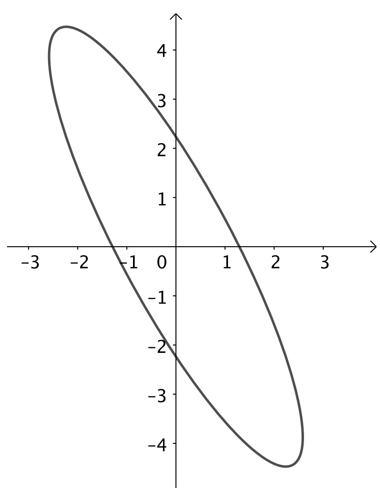
$(3x+2y)\frac{dy}{dx} = -6x-3y$

$\frac{dy}{dx} = \frac{-6x-3y}{3x+2y}$

other correct forms for $\frac{dy}{dx}$:

$\frac{dy}{dx} = \frac{6x+3y}{-3x-2y} = -\frac{6x+3y}{3x+2y} = \frac{(-3)(2x+y)}{3x+2y}$

(b) [5 pts] Determine all values of x for which the point (x, y) on the curve $3x^2 + 3xy + y^2 = 5$ has a horizontal tangent line. The graph of the curve is provided below for your reference.



$6x + 3y = 0$

$6x = -3y$

$-2x = y$

$3x^2 + 3x(-2x) + (-2x)^2 = 5$

$3x^2 - 6x^2 + 4x^2 = 5$

$x^2 = 5$

$\boxed{x = \pm\sqrt{5}}$

6. Determine the following indefinite integrals.

(a) [6 pts] $\displaystyle\int \left(x^3 - 5x + 7\right)\, dx = \boxed{\frac{1}{4}x^4 - \frac{5}{2}x^2 + 7x + C}$

(b) [6 pts] $\displaystyle\int \frac{(\ln(x)+4)^{10}}{x}\, dx = \int u^{10}\,du = \frac{1}{11}u^{11} + C = \boxed{\frac{1}{11}(\ln(x)+4)^{11} + C}$

$u = \ln(x) + 4$

$du = \frac{1}{x}\,dx$

(c) [6 pts] $\displaystyle\int \frac{\sin(t)}{1 - 2\cos(t)}\, dt = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln\left|1-2\cos(t)\right| + C}$

$u = 1 - 2\cos(t)$

$du = 2\sin(t)\,dt$

7. Evaluate the following definite integrals.

(a) [6 pts] $\displaystyle\int_0^1 \left(e^x - \frac{3}{1+x^2}\right)\, dx$ $= \left[e^x - 3\arctan(x)\right]_0^1$

$= (e^1 - 3\arctan(1)) - (e^0 - 3\arctan(0))$

$= e - \frac{3\pi}{4} - 1$

(b) [8 pts] $\displaystyle\int_0^3 f(x)\, dx$, where $f(x)$ is the function given by $f(x) = \begin{cases} \sin(x) & 0 \le x < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} \le x \le 3 \end{cases}$

$\int_0^{\pi/2} \sin(x) = \left[-\cos(x)\right]_0^{\pi/2} = (-\cos(\pi/2)) - (-\cos(0)) = 0 - (-1) = 1$

$\int_{\pi/2}^{3} 1\, dx = \left[x\right]_{\pi/2}^{3} = 3 - \pi/2$

$\int_0^3 f(x)\, dx = 1 + 3 - \pi/2 = \boxed{4 - \pi/2}$

8. [10 pts] Use calculus techniques to determine the x-coordinates of the local (relative) extrema for the function below. Be sure to label each extremum as a local (relative) maximum or as a local (relative) minimum.

$$f(x) = x^6 - 4x^3$$

domain: $(-\infty, \infty)$

$f'(x) = 6x^5 - 12x^2$
$= 6x^2(x^3 - 2)$

$x = 0, \sqrt[3]{2}$

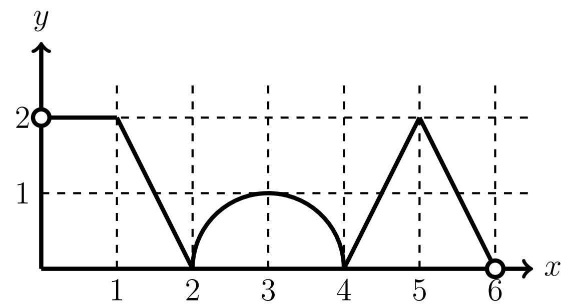
1st derivative test:

	$(-\infty, 0)$	$(0, \sqrt[3]{2})$	$(\sqrt[3]{2}, \infty)$
f'	$f'(-1) = 6(-3)$	$f'(1) = 6(-1)$	$f'(2) = 24(6) = 144$
sign f'	$-$	$-$	$+$
beh. f	dec $\searrow$	dec $\searrow$	inc $\nearrow$

rel min

Note The second derivative test is inconclusive at $x=0$.

9. The graph of $y = f(x)$ is given below and consists of line segments and circle arcs. The domain of f is $(0, 6)$.



(a) [5 pts] Determine all values of x in $(0, 6)$ for which $f'(x) > 0$. Write your answer using interval notation.

(2,3), (4,5)

(b) [5 pts] Determine all values of x in $(0, 6)$ for which $f'(x)$ is undefined.

1, 2, 4, 5

(c) [5 pts] Estimate $f'(1.5)$.

-2

(d) [5 pts] Determine whether $f''(3)$ is positive, negative, or zero. (Write "positive," "negative," or "zero." No explanation is needed.)

negative

(e) [5 pts] Determine $\displaystyle\int_0^6 f(x)dx$.

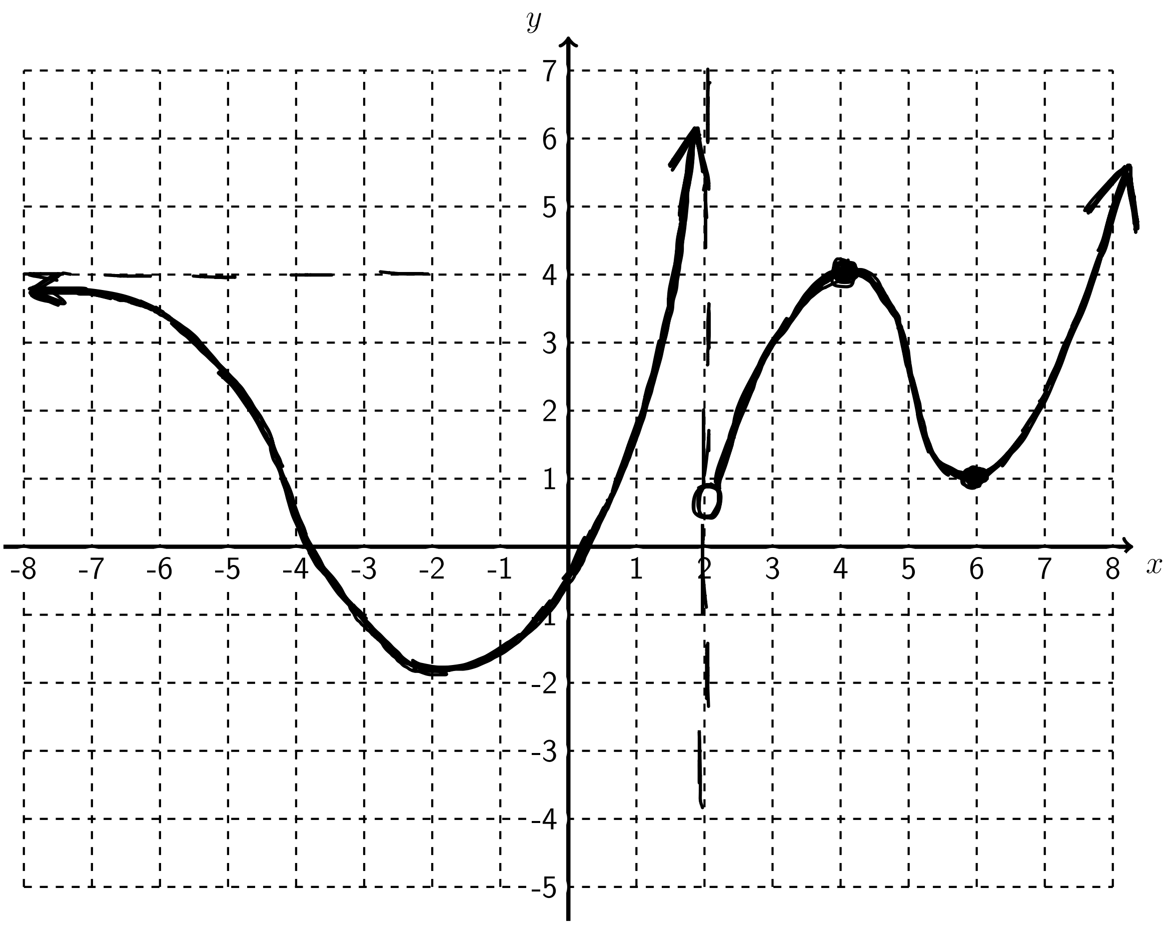
$2 + 1 + \frac{\pi(1)^2}{2} + 2 = 5 + \frac{\pi}{2}$

10. [14 pts] On the axes provided below, sketch a graph of a function $y = f(x)$ which meets the following criteria:

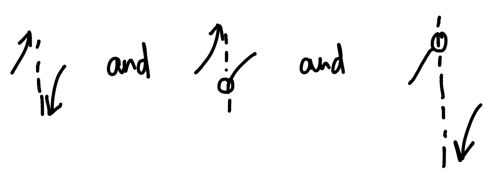
- Its domain is $(-\infty, 2) \cup (2, \infty)$.
- The line $x = 2$ is a vertical asymptote to the graph of $y = f(x)$.
- The function f is continuous at every point in its domain.
- The sign chart for the derivative of f is the following:

interval	$(-\infty, -2)$	$(-2, 2)$	$(2, 4)$	$(4, 6)$	$(6, \infty)$
sign of f'	$-$	$+$	$+$	$-$	$+$

- The function satisfies $f(4) = 4$ and $f(6) = 1$.
- $\lim_{x\to\infty} f(x) = \infty$
- The line $y = 4$ is a horizontal asymptote to the graph of $y = f(x)$.



Near x=2, these are all correct:



11. [10 pts] Determine all values of c for which the function below is continuous on $(-\infty, \infty)$. Use the limit definition of continuity to explain your answer.

$$f(x) = \begin{cases} \dfrac{9-x^2}{3-x}, & x < 3 \\ c, & x \geq 3 \end{cases}$$

Note: f is already continuous everywhere except $x=3$

To make f continuous at $x=3$, we need $\lim_{x\to 3} f(x) = f(3)$.

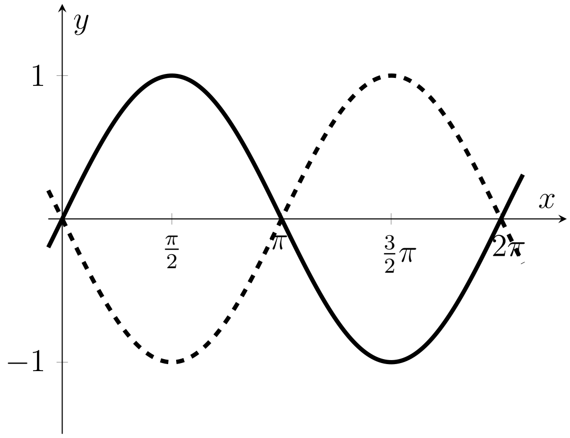
$f(3) = c$

$\lim_{x\to 3^+} f(x) = c$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{9-x^2}{3-x} = \lim_{x\to 3^-} \frac{(3+x)(3-x)}{3-x} = \lim_{x\to 3^-} (3+x) = 6$

To make f continuous at $x=3$, we need $c=6$.

12. [10 pts] Find the total area of the region enclosed by the curves $y = \sin(x)$ (solid) and $y = -\sin(x)$ (dashed) from $x = 0$ to $x = 2\pi$. The relevant graphs are provided below for your reference.



$$\int_0^{\pi}(\sin(x)-(-\sin(x)))dx = \int_0^{\pi} 2\sin(x)dx = \left[-2\cos(x)\right]_0^{\pi} = (-2\cos(\pi)) - (-2\cos(0)) = 2+2=4$$

To get total area enclosed, can use any of the methods below:

① $\int_0^{\pi}(\sin(x) - (-\sin(x)))dx = 4$ using above steps

$$\int_{\pi}^{2\pi}(-\sin(x) - \sin(x))dx = \int_{\pi}^{2\pi} -2\sin(x)\,dx = \left[2\cos(x)\right]_{\pi}^{2\pi} = 2-(-2)=4$$

Total: $4+4=\boxed{8}$

② $\int_0^{\pi}(\sin(x)-(-\sin(x)))dx = 4$ using above steps

Total area: $2(4)=\boxed{8}$

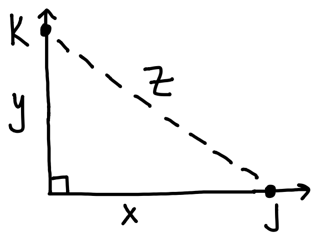
③ $\int_0^{\pi}\sin(x)dx = \left[-\cos(x)\right]_0^{\pi} = 1--1 = 2$

Total area $4(2)=\boxed{8}$

④ $\int_0^{\pi/2}\sin(x)dx = \left[-\cos(x)\right]_0^{\pi/2} = 0-(-1) = 1$

Total area: $8(1)=\boxed{8}$

13. [15 pts] Karina and Juan both leave an intersection at the same time, both driving on long, straight roads. Karina drives north at a speed of 50 miles per hour, and Juan drives east at 60 miles per hour. When 1.5 hours have elapsed since they left the intersection, how fast is the distance between Karina and Juan changing? Use calculus to justify your answer.



$\frac{dy}{dt} = 50$ mph

$\frac{dx}{dt} = 60$ mph

goal: $\frac{dz}{dt}$ when $t = 1.5$

$z^2 = x^2 + y^2$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}$

$t = 1.5$ hours

$x = 60(1.5) = 90$ miles

$y = 50(1.5) = 75$ miles

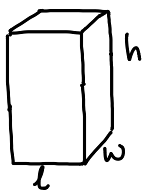
$z = \sqrt{90^2 + 75^2}$

$\frac{dz}{dt} = \boxed{\frac{90(60) + 75(50)}{\sqrt{90^2+75^2}} \text{ mph}}$

Note: $\frac{dz}{dt}$ is approximately 78 mph. Also provide the exact answer (shown in the box above) if you give an approximate answer like 78 mph.

14. You plan to make a rectangular box having an open top. Each face of the box is a rectangle, and the top face of the box is missing so that the box is open. For the base of the box, you want the length to be twice the width; you also want the total surface area of the resulting box to be 200 square inches.

_______ (a) [12 pts] Let w represent the width of the box in inches. Determine a formula $V(w)$ for the total volume of the box as a function of w alone. Your final answer should include the variable w and can not include any other variables.



$l = 2w$

$V = lwh$

$200 = lw + 2wh + 2lh$

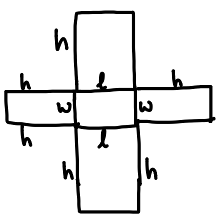
constraint equation

$200 = (2w)(w) + 2wh + 2(2w)h$

$200 = 2w^2 + 2wh + 4wh$

$200 = 2w^2 + 6wh$

$\frac{200 - 2w^2}{6w} = h$ refined constraint equation



$V = (2w)(w)\left(\frac{200 - 2w^2}{6w}\right)$

$V = \frac{2w(200 - 2w^2)}{6} = \frac{1}{3}(200w - 2w^3)$

All of these are valid ways to write your final answer.

_______ (b) [4 pts] Suppose you want to determine the dimensions of the box that will result in the maximum possible volume. Determine an appropriate domain for $V(w)$. Briefly explain your answer. (Note: We will not actually maximize $V(w)$ in this problem.)

$(0, 10)$ or $(0, 10]$

$w = 0$ violates constraint equation(s); w can't be negative

Larger w corresponds to smaller h; set $h = 0$:

$0 = \frac{200 - 2w^2}{6w}$

$0 = 200 - 2w^2$

$2w^2 = 200$

$w = 10$

15. Suppose you run a factory that processes raw materials in two possible ways.

- Using method A, you can process x tons of material at a cost of x^2 dollars.
- Using method B, you can process y tons of material at a cost of $10y$ dollars.

You want to process a total of 200 tons of material, so your process is constrained by the equation $x + y = 200$. The overall cost (in dollars) of processing the two materials is

$$C = x^2 + 10y.$$

You can process a partial ton of material using each method, so x and y do not have to be whole numbers. You plan to minimize the overall cost.

(a) [4 pts] Determine the overall cost C **as a function of** x. (The variable y should not be used in your answer.)

$x + y = 200$

$y = 200 - x$

$C = x^2 + 10y$

$C = x^2 + 10(200-x) = x^2 + 2000 - 10x$

$= x^2 - 10x + 2000$

These are all correct ways to write your final answer.

(b) [4 pts] Determine an appropriate domain for your cost function C from part (a), and explain briefly.

$x=200$, using method A exclusively

$[0, 200]$

$x=0$, using method B exclusively

[This problem continues on the next page.]

[Problem 15, continued]

(c) [12 pts] Use calculus techniques to determine the value of x which results in the minimum overall cost. Write a sentence, using appropriate units, to summarize your answer.

$C = x^2 - 10x + 2000$

$[0, 200]$

$C' = 2x - 10$
$x = 5$

Extreme value theorem/closed interval method:

x	C
0	2000
5	$25 - 50 + 2000 = 1975$ ← min cost
200	$40000 - 2000 + 2000 = 40000$

When $x = 5$ tons, the overall cost is minimized.

This page is extra space for work. **Do not detach this page.** If you want us to consider the work on this page, you should print your name, instructor and class meeting time below. For the problems where you want us to look at this work, please write "see last page" next to your work on the problem page so that we know to look here.

Name (print): ______________________ Instructor (print): ______________________

Class Meeting Time: ______________