Qualifying Exam: Complex Analysis — Fall 2020

1. Let $n \geq 2$ be an integer. Show that $2^{n-1} \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = n$.

   [Hint: Use n-th roots of unity i.e., solutions of $z^n - 1 = 0$]

2. Expand $\dfrac{1}{1-z^2} + \dfrac{1}{z-3}$ in a series of the form $\sum_{-\infty}^{\infty} a_n z^n$ so it converges for (a) $|z| < 1$, (b) $1 < |z| < 3$; and (c) $|z| > 3$.

3. Let $a \in \mathbb{R}$ with $0 < a < 3$. Evaluate $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^3} dx$.

4. Let $\mathbb{D} := \{z : |z| < 1\}$ denote the open unit disk. Suppose that $f(z) : \mathbb{D} \to \mathbb{D}$ is holomorphic, and that there exists $a \in \mathbb{D} \setminus \{0\}$ such that $f(a) = f(-a) = 0$.

   (a) Prove that $|f(0)| \leq |a|^2$.

   (b) What can you conclude when $|f(0)| = |a|^2$?

5. Consider the function $f(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ for $z \in \mathbb{C} \setminus \{0\}$. Let $\mathbb{D}$ denote the open unit disc.

   (a) Show that f is one-to-one on the punctured disc $\mathbb{D} \setminus \{0\}$. What is the image of the circle $|z| = r$ under this map when $0 < r < 1$?

   (b) Show that f is one-to-one on the domain $\mathbb{C} \setminus \bar{\mathbb{D}}$. What is the image of this domain under this map?

   (c) Show that there exists a map $g : \mathbb{C} \setminus [-1, 1] \to \mathbb{D} \setminus \{0\}$ such that $(g \circ f)(z) = z$ for all $z \in \mathbb{D} \setminus \{0\}$. Describe the map g by an explicit formula.

6. Suppose that U is a bounded, open and simply connected domain in $\mathbb{C}$ and that $f(z)$ is a complex-valued non-constant continuous function on $\bar{U}$ whose restriction to U is holomorphic.

   (a) Prove the maximum modulus principle by showing that if $z_0 \in U$, then
   $$|f(z_0)| < \sup\{|f(z)| : \ z \in \partial U\}.$$

   (b) Show furthermore that if $|f(z)|$ is constant on ∂U, then $f(z)$ has a zero in U (i.e., there exists $z_0 \in U$ for which $f(z_0) = 0$).

7. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic and $f(0) = 0$. Let $n \geq 1$, and define the function $f_n(z)$ to be the n-th composition of f with itself; more precisely, let
   $$f_1(z) := f(z), \ f_2(z) := f(f(z)), \text{ in general } f_n(z) := f(f_{n-1}(z)).$$
   Suppose that for each $z \in \mathbb{D}$, $\lim_{n\to\infty} f_n(z)$ exists and equals to $g(z)$. Prove that either $g(z) \equiv 0$ or $g(z) = z$ for all $z \in D$.