VERSION B

MATH 2250 FINAL EXAMINATION
May 5, 2015

Directions: You have 3 hours to complete 16 questions worth 200 total points. You may use only a TI30 calculator. If you use a calculator, give decimal answers correct to three (3) decimal places. No cell phones, laptops, tablets, books, or notes are allowed at or near your seat.

On all questions requiring calculation of any sort, you must show your algebra and calculus work in order to earn credit. You do not need to simplify expressions, although you may need to in order to complete the next steps of a problem.

My name written below confirms that I have read and understood the UGA Culture of Honesty. I will be academically honest in all of my academic work and will not tolerate academic dishonesty of others. I will complete this test in strict compliance with these policies.

Name: Solutions

Instructor's name: ____________________

Room/Seat Number: ____________________

#	Score	Points
1		14
2		17
3		14
4		12
5		10
6		12
7		15
8		15
9		12
10		6
11		10
12		15
13		8
14		15
15		15
16		10
Total		200

(1) (14 points) Compute the following limits. On parts a, b, you MUST show work. For c, d, work is not required.

(a) $\lim_{x \to -2} \frac{x+2}{x^2+4} = \frac{-2+2}{4+4} = \frac{0}{8} = \boxed{0}$

(b) $\lim_{x \to 0} \frac{e^{3x}-1}{\sin x} \overset{LH}{=} \lim_{x \to 0} \frac{3e^{3x}}{\cos(x)} = \frac{3}{1} = \boxed{3}$

IF $\frac{0}{0}$

(c) $\lim_{x \to 4^-} \frac{1}{\sqrt{4-x}} = \infty$

numerator is 1

denominator $\to 0$ and is positive

(d) $\lim_{x \to \infty} \frac{x^2-1}{8x^2-8x} \cdot \frac{(1/x^2)}{(1/x^2)} = \lim_{x \to \infty} \frac{1-1/x^2}{8-8/x} = \frac{1-0}{8-0} = \boxed{1/8}$

(2) (17 points) Compute $\frac{dy}{dx}$ for the following functions:

(a) $y = \frac{x^5}{3} - \frac{17}{x^2} - \csc x$

$$\frac{dy}{dx} = \frac{5x^4}{3} + 34x^{-3} + \csc(x)\cot(x)$$

(b) $y = \frac{(2x+5)^3}{e^x+7}$

$$\frac{dy}{dx} = \frac{(e^x+7)\cdot 3(2x+5)^2 \cdot 2 - (2x+5)^3 \cdot e^x}{(e^x+7)^2}$$

(c) $y = \sin^{-1}(x^2 \cos x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(x^2\cos(x))^2}} \cdot \left[x^2 \cdot -\sin(x) + \cos(x)\cdot 2x\right]$$

(d) $y = (x^2+1)^{4x}$ * logarithmic differentiation *

$$\ln(y) = \ln\left[(x^2+1)^{4x}\right]$$

$$\ln(y) = 4x\ln(x^2+1)$$

$$\frac{1}{y}\frac{dy}{dx} = 4x \cdot \frac{2x}{x^2+1} + \ln(x^2+1)\cdot 4$$

$$\frac{dy}{dx} = y\left(\frac{8x^2}{x^2+1} + 4\ln(x^2+1)\right)$$

$$\frac{dy}{dx} = (x^2+1)^{4x}\left(\frac{8x^2}{x^2+1} + 4\ln(x^2+1)\right)$$

(3) (14 points) Let $f(x) = \dfrac{1}{3x-2}$

(a) Use the **definition** of the derivative to compute $f'(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{3(x+h)-2} - \frac{1}{3x-2}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{3x+3h-2} - \frac{1}{3x-2}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{3x+3h-2}\cdot\frac{3x-2}{3x-2} - \frac{1}{3x-2}\cdot\frac{3x+3h-2}{3x+3h-2}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{3x-2}{(3x+3h-2)(3x-2)} - \frac{3x+3h-2}{(3x-2)(3x+3h-2)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{3x-2-3x-3h+2}{(3x+3h-2)(3x-2)}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\left(\frac{-3h}{(3x+3h-2)(3x-2)}\right)$$

$$= \lim_{h\to 0} \frac{-3}{(3x+3h-2)(3x-2)} = \boxed{\frac{-3}{(3x-2)^2}}$$

(b) Find the equation of the line tangent to $f(x)$ at $x = 2$.

$$f(2) = \frac{1}{3\cdot 2-2} = \frac{1}{4}$$

$$f'(2) = \frac{-3}{(3\cdot 2-2)^2} = \frac{-3}{16}$$

$$\boxed{y - \frac{1}{4} = \frac{-3}{16}(x-2)}$$

(4) (12 points) The "figure-eight curve" given by the equation $y^4 = y^2 - x^2$ is sketched below.

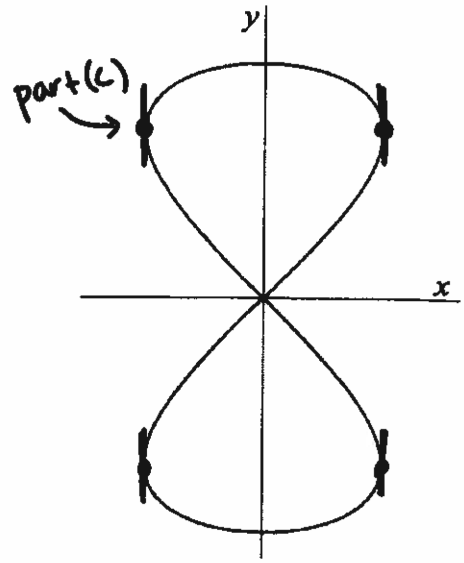
(a) Find $\frac{dy}{dx}$.

$4y^3\frac{dy}{dx} = 2y\frac{dy}{dx} - 2x$

$4y^3\frac{dy}{dx} - 2y\frac{dy}{dx} = -2x$

$(4y^3 - 2y)\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{4y^3-2y} = \frac{x}{y-2y^3}$



(b) On the graph above, sketch ALL the vertical tangent lines.

(c) Find the coordinates of any ONE of the points at which the curve has a vertical tangent. Circle your final answer.

$y - 2y^3 = 0$

$y(1-2y^2) = 0$

$y = 0 \quad 1 - 2y^2 = 0$

$1 = 2y^2$

$\frac{1}{2} = y^2$

$y = \pm\frac{1}{\sqrt{2}}$

go back to original:

$y^4 = y^2 - x^2$

$y = 0 \Rightarrow 0 = -x^2$ so $x = 0$

(0,0) does not have a vertical tangent (see graph)

$y = \pm\frac{1}{\sqrt{2}} \Rightarrow \frac{1}{4} = \frac{1}{2} - x^2$

$x^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

$x = \pm\frac{1}{2}$

one point with a vertical tangent is

$\boxed{\left(-\frac{1}{2}, \frac{1}{\sqrt{2}}\right)}$ (There are 3 more.)

(5) (10 points)

(a) State the Mean Value Theorem.

If f is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one number c in (a,b) satisfying

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

(b) Given that a function $f(x)$ satisfies the hypotheses of the Mean Value Theorem for the interval $[-3,2]$, and we have data $f(2) = -11$ and $f'(x) \leq 4$ for all x in $[-3,2]$, what is the smallest possible value of $f(-3)$?

$$\frac{f(2)-f(-3)}{2-(-3)} = f'(c) \leftarrow \text{at most } 4$$

$$\frac{-11-f(-3)}{5} \leq 4$$

$$-11 - f(-3) \leq 20$$

$$-f(-3) \leq 31$$

$$f(-3) \geq -31$$

$$\boxed{-31}$$

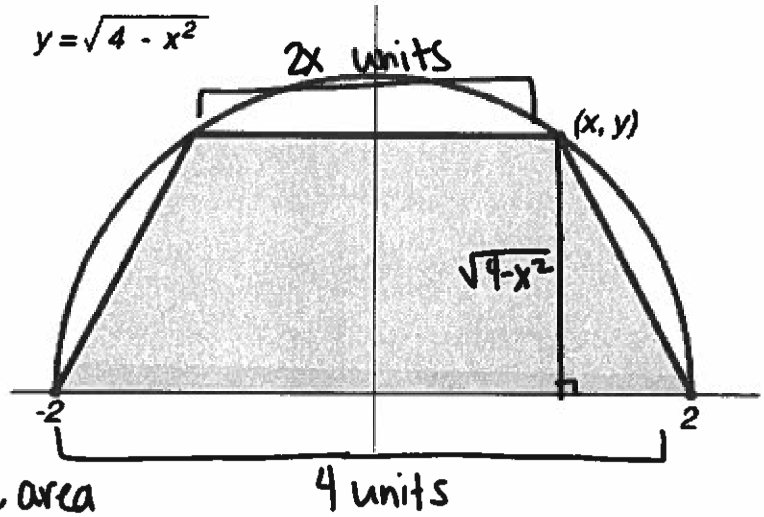
(6) (12 points) Work out the following antiderivatives using any technique that you like.

(a) $\int \sec(7x+2)\tan(7x+2)\,dx = \frac{1}{7}\sec(7x+2) + C$

(b) $\int \frac{x}{(x^2+4)^{17}}\,dx = \int x(x^2+14)^{-17}\,dx = \frac{-1}{32}(x^2+14)^{-16} + C$

(7) (15 points) Consider a trapezoid sketched in the coordinate plane below. Two vertices of the trapezoid are located at $(-2, 0)$ and $(2, 0)$ and the other two lie on the semicircle $y = \sqrt{4 - x^2}$. What is the maximum possible area of such a trapezoid? (Note: the area of any trapezoid with bases b_1 and b_2 and height h is $\frac{1}{2}h(b_1 + b_2)$)

Be sure to write a summary sentence, and provide justification that you've solved the problem.



goal: maximize area

$$A = \frac{1}{2}\left(\sqrt{4-x^2}\right)(4 + 2x), \text{ domain } [0,2]$$

$$A' = \frac{1}{2}\sqrt{4-x^2} \cdot 2 + (4+2x) \cdot \frac{1}{2} \cdot \frac{1}{2}(4-x^2)^{-1/2} \cdot -2x$$

$$= \sqrt{4-x^2} + \frac{(4+2x)(-x)}{2\sqrt{4-x^2}} = \frac{\sqrt{4-x^2} \cdot 2\sqrt{4-x^2}}{2\sqrt{4-x^2}} + \frac{-4x-2x^2}{2\sqrt{4-x^2}}$$

$$= \frac{2(4-x^2)-4x-2x^2}{2\sqrt{4-x^2}} = \frac{8-2x^2-4x-2x^2}{2\sqrt{4-x^2}} = \frac{-4(x^2+x-2)}{2\sqrt{4-x^2}} = \frac{-4(x+2)(x-1)}{2\sqrt{4-x^2}}$$

critical numbers:

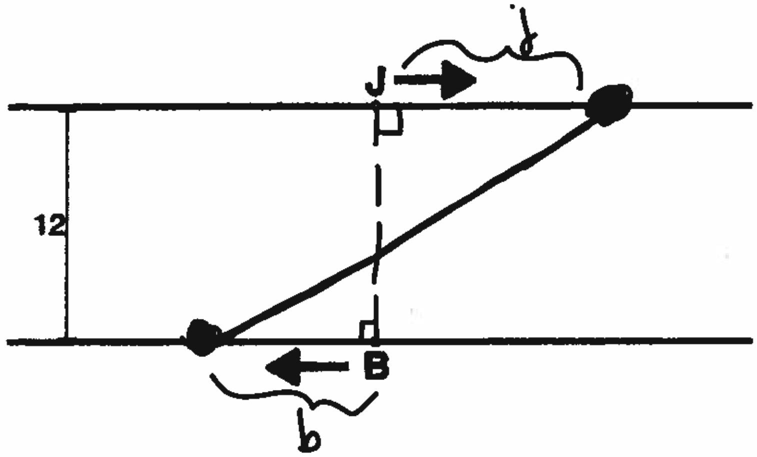
A' dne: none in domain $A'=0$: $\cancel{x=-2}$, $x=1$

x	A
0	$\frac{1}{2}\sqrt{4} \cdot 4 = 4$
1	$\frac{1}{2}\sqrt{3} \cdot 6 = 3\sqrt{3} \approx 5.19615$
2	$\frac{1}{2}\sqrt{0} \cdot 8 = 0$

$\longleftarrow$ The maximum possible area is $3\sqrt{3}$ square units.

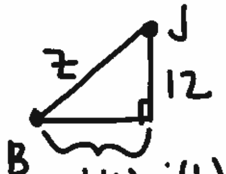
(8) (15 points) Jane and Bob are standing opposite each other on two straight footpaths, which are situated 12 ft apart. At the same instant, Jane starts walking East at a rate of 2 ft/sec, and Bob starts jogging West at a rate of 3 ft/sec. How fast is the distance between Jane and Bob changing 4 seconds later?

Be sure to write a summary sentence with correct units.



$j'(t) = 2$ ft/s (constant), $b(0)=0 \Rightarrow j(t) = 2t$ feet

$b'(t) = 3$ ft/s (constant), $b(0)=0 \Rightarrow b(t) = 3t$ feet



$b(t)+j(t) = 5t$

$$z^2 = (5t)^2 + (12)^2 = 25t^2 + 144$$

$$2z\frac{dz}{dt} = 50t$$

$$\frac{dz}{dt} = \frac{25t}{z}$$

$t=4$ so $z^2 = 25 \cdot 16 + 144$

$z = \sqrt{544}$

$$\frac{dz}{dt} = \frac{100}{\sqrt{544}} \text{ ft/sec}$$

The distance between Jane and Bob is increasing at a rate of $\frac{100}{\sqrt{544}}$ ft/sec.

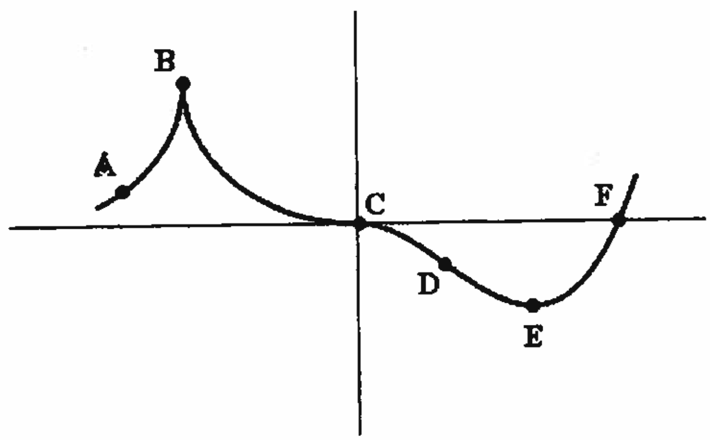
(9) (12 points) Examine the graph of the function $h(x)$ below, with points A,B,C,D,E,F marked. (CIRCLE all of the points which meet the required conditions, no explanations are necessary)

(a) Which of the points A,B,C,D,E,F are critical points for h?

(b) At which of the points A,B,C,D,E,F is $h' = 0$ and $h'' > 0$?

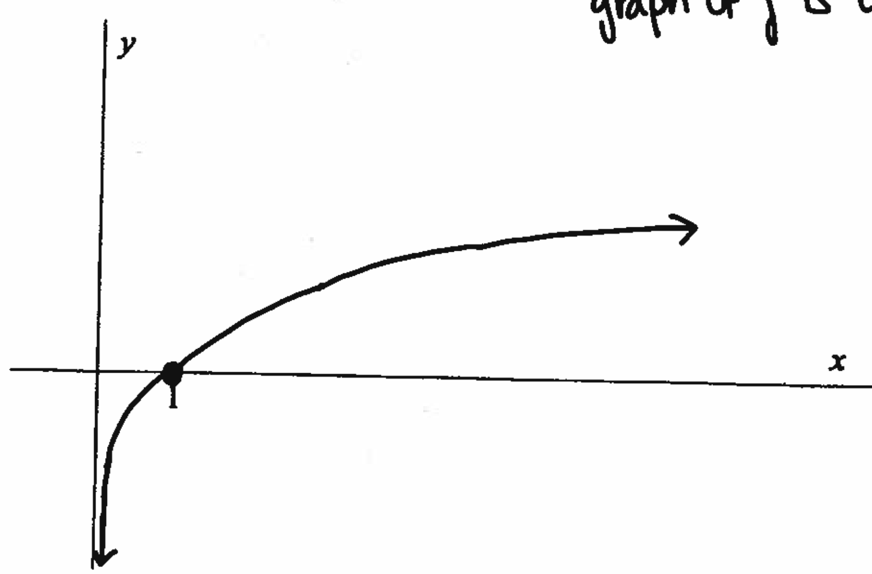
(c) At which of the points A,B,C,D,E,F is $h' > 0$ and $h'' > 0$?

(d) At which of the points A,B,C,D,E,F does h'' change sign?



(10) (6 points) On the axes provided below, Draw the graph of a single function $j(x)$ that satisfies ALL of these conditions:
The domain of j is $(0, \infty)$ The y-axis is a vertical asymptote $j(1) = 0$
The function j increases at a decreasing rate throughout the domain

↳ $j' > 0$ ↳ j' is decreasing
graph of j is concave down



(11) (10 points) A certain (unknown) function $g(x)$ has derivatives $g'(x) = \sqrt{x^2+12}$ and $g''(x) = \dfrac{x}{\sqrt{x^2+12}}$. It is known that $g(2) = 7$.

(a) Find the linearization of $g(x)$ at $x = 2$.

$g(2)=7$

$g'(2)=\sqrt{16}=4$

$L(x)=g(2)+g'(2)(x-2)$

$L(x)=7+4(x-2)$

(b) Use your linearization to **estimate** $g(1.9)$

$g(1.9)\approx L(1.9)$

$L(1.9)=\boxed{7+4(1.9-2)}$

(c) Explain why you know that your estimate is an UNDERestimate.

$g''(2)=\frac{1}{2}$ so the graph of g is concave <u>up</u> near $x=2$

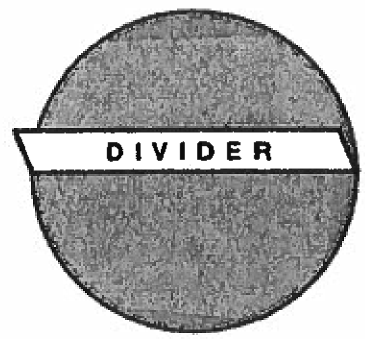
Since g is concave up, the tangent line lies below the graph, so its y-values will under-estimate the y-values on the graph of g.

(12) (15 points) An inexpensive cookie serving tray is made from a cardboard circular disk. A rectangular cardboard divider sticks up from the disk with an unknown height. The divider lies across a diameter of the disk. (See the figure below.) In total, 48π square inches of cardboard is used to make this tray (disk and divider). We want to fill this tray with cookies, all the way to the top of the divider. Find the radius of the disk and the divider height that will maximize the total volume of the cylindrical shape made by the full tray.

Be sure to write a summary sentence, and provide justification that you've solved the problem.

goal: maximize volume

$V = \pi r^2 h$



Constraint: $48\pi = \pi r^2 + 2rh$

$$\frac{48\pi - \pi r^2}{2r} = h$$

$$V = \pi r^2 \left(\frac{48\pi - \pi r^2}{2r}\right) = \pi r\left(\frac{48\pi - \pi r^2}{2}\right)$$

$$V = \frac{\pi^2}{2} \cdot r(48 - r^2) = \frac{\pi^2}{2}(48r - r^3)$$

domain: $r > 0$, $48\pi - \pi r^2 = 0$ gives upper cutoff for r

$\pi(48 - r^2) = 0$ domain: $(0, \sqrt{48})$

$r = \sqrt{48}$

$V' = \frac{\pi^2}{2}(48 - 3r^2)$ critical numbers: $48 - 3r^2 = 0$

$48 = 3r^2$

$16 = r^2$

$r = 4$

Second derivative test:

$V'(4) = 0$

$V''(x) = \frac{\pi^2}{2}(-6r)$ so $V''(4) = \frac{\pi^2}{2}(-24) < 0$

Therefore V has a local max at $r=4$. Since $r=4$ is the only critical number in the domain $(0, \sqrt{48})$, the local max is also an absolute max. The dimensions are $r = 4$ in. and $h = \frac{48\pi - 16\pi}{8} = \frac{32\pi}{8} = 4\pi$ in.

(13) (8 points)

(a) Use part one of the Fundamental Theorem of Calculus to compute:

$$\frac{d}{dx}\int_1^x \sqrt{t^2+\sin t}\,dt =$$

$$\sqrt{x^2+\sin(x)}$$

(b) Use part two of the Fundamental Theorem of Calculus to compute:

$$\int_{\pi/6}^{\pi/4} \csc^2(t)\,dt =$$

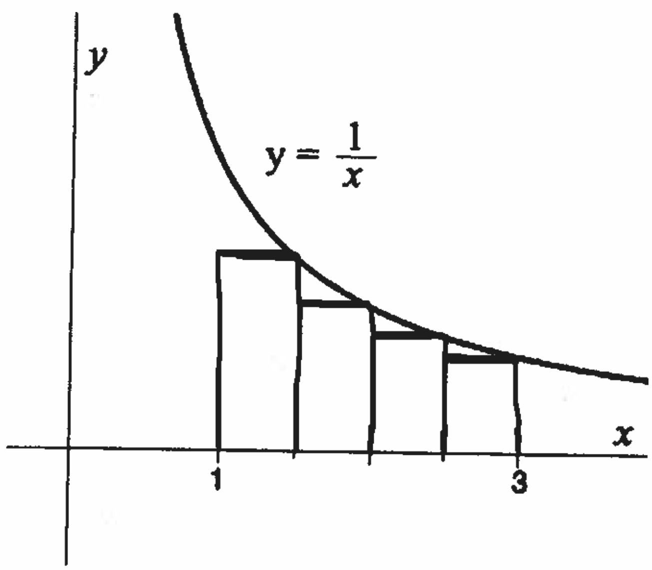
$$\Big[-\cot(t)\Big]_{t=\pi/6}^{t=\pi/4} = -\cot(\pi/4) + \cot(\pi/6)$$

$$= -1+\sqrt{3}$$

(14) (15 points)

(a) Evaluate $\int_1^3 \frac{1}{x}\,dx. = \left[\ln|x|\right]_{x=1}^{x=3} = \ln|3| - \ln|1| = \boxed{\ln(3)}$

(b) On the graph of $y = \dfrac{1}{x}$ below, sketch in *four right endpoint* rectangles to approximate the value you computed in part a.



(c) Now compute your area estimate using the four right endpoint rectangles.

Subintervals:

$[1, \frac{3}{2}]\ [\frac{3}{2}, 2]\ [2, \frac{5}{2}]\ [\frac{5}{2}, 3]$

$\ast \quad \ast \quad \ast \quad \ast$

$\text{area} \approx f(\frac{3}{2})\cdot\frac{1}{2} + f(2)\frac{1}{2} + f(\frac{5}{2})\cdot\frac{1}{2} + f(3)\cdot\frac{1}{2}$

$= \frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{2}} + \frac{1}{2}\cdot\frac{1}{2} + \frac{\cancel{2}}{5}\cdot\frac{1}{\cancel{2}} + \frac{1}{3}\cdot\frac{1}{2} = \boxed{\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}}$

(15) (a) (5 points) Short answer, no work required.
We represent the rate of change of the length of a fish t days after birth by $\ell'(t)$, where the units of $\ell'(t)$ are cm/day. The length of the fish at birth ($t = 0$) is 2 cm. What does the number $2 + \int_0^{14} \ell'(t)\,dt$ represent? Answer in a sentence, with units.

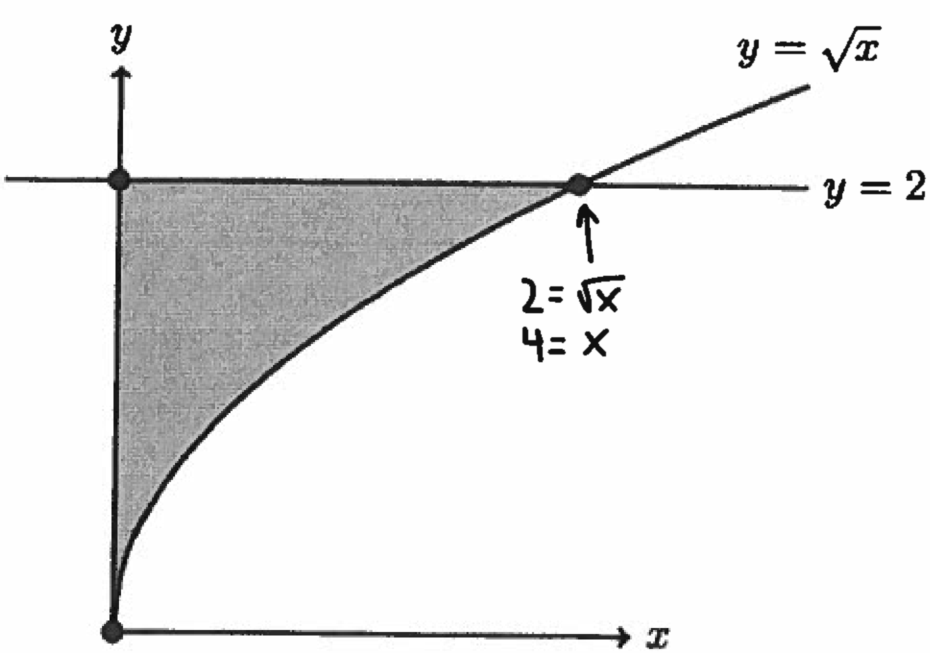
It represents the length of the fish in centimeters when the fish is 14 days old, since

$$\ell(14) - \ell(0) = \int_0^{14} \ell'(t)\,dt$$

$$\ell(14) = \ell(0) + \int_0^{14} \ell'(t)\,dt = 2 + \int_0^{14} \ell'(t)\,dt.$$

(b) (10 points) Consider the shaded region pictured below, bounded by the y-axis, the line $y = 2$ and $y = \sqrt{x}$. Write down and evaluate a definite integral(s) to compute the area of the shaded region.

$$\int_0^4 (2 - \sqrt{x})\,dx = \left[2x - \tfrac{2}{3}x^{3/2}\right]_0^4 = 8 - \tfrac{2}{3}(4)^{3/2} - 0 = 8 - \tfrac{2}{3}(\sqrt{4})^3$$
$$= 8 - \tfrac{2}{3}\cdot 8$$
$$= \boxed{8/3}$$



(16) (10 points) Dr. Teddy is wearing a bungee cord and a rocket pack. He jumps off a bridge at time $t = 0$. We measure Dr. Teddy's position function in feet ABOVE the bridge, t minutes after jumping. Dr. Teddy's acceleration upward after t minutes is given by the function:

$$a(t) = 6\sin(3t) \text{ ft/min}^2$$

(a) If the initial velocity is $v(0) = -1$ ft/min, then find Dr. Teddy's velocity function $v(t)$ in ft/min.
HINT: Pay close attention to the initial condition $v(0) = -1$.

$V(t) = -2\cos(3t) + C$

$V(0) = -2\cos(0) + C$

$-1 = -2 + C$

$1 = C$

$\boxed{V(t) = -2\cos(3t) + 1}$

(b) Given that $s(0) = 0$, find Dr. Teddy's position function $s(t)$ in ft.

$S(t) = \frac{-2}{3}\sin(3t) + t + D$

$S(0) = -\frac{2}{3}\sin(0) + 0 + D$

$0 = D$

$\boxed{S(t) = \frac{-2}{3}\sin(3t) + t}$

(c) When is the first time that Dr. Teddy changes direction?

possible direction change when $V = 0$

$-2\cos(3t) + 1 = 0$

$-2\cos(3t) = -1$

$\cos(3t) = 1/2$

$3t = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots$

$t = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \ldots$ Ck: $a\left(\frac{\pi}{9}\right) = 6\sin\left(\frac{\pi}{3}\right) > 0$

graph of $S(t)$ near $\frac{\pi}{9}$:

direction change

smallest t ("first time") is $\boxed{t = \frac{\pi}{9}}$