University of Georgia **Math 2250**
Department of Mathematics **Final Exam** Fall 2015

By providing my signature below I acknowledge that this is my work, and I did not get any help from anyone else:

Name (sign): Solutions Name (print): ____________

Student Number: ____________

Instructor's Name: ____________ Meeting Time: ____________

Problem Number	Points Possible	Points Made
1	60	
2	20	
3	5	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
Total:	185	

- If you need extra space use the last page.
- Please show your work. **An unjustified answer may receive little or no credit.**
- Your test must be **neat**. We will take off points for sloppiness.
- If it is determined that you copied the work of another student you will receive no points for this test.
- The total number of points that is assigned for each problem is given. This is the *total* number of points for a problem and *not* the number of points for each subproblem.
- Please turn off your mobile phone.
- If you use your calculator to do any operation, write down the steps you take *verbatim*.

1. Determine the derivatives of each of the following functions:

(a) [10 pts] $f(x) = 4x + 8x^{0.2}$

$f'(x) = 4 + 1.6x^{-0.8}$

(b) [10 pts] $g(x) = \dfrac{x}{1+x} = x(1+x)^{-1}$

$g'(x) = \dfrac{(1+x)\cdot 1 - x(1)}{(1+x)^2}$ or $g'(x) = x(-(x+1)^{-2}) + (x+1)^{-1} \cdot 1$

(c) [10 pts] $h(x) = (x^2+1)^2 \cdot \sin(2x)$

$h'(x) = (x^2+1)^2 \cdot 2\cos(2x) + \sin(2x) \cdot 2(x^2+1) \cdot 2x$

(d) [10 pts] $u(x) = \frac{1}{\sqrt[3]{x}} + e^{-x^2+1}$ $= x^{-1/3} + e^{-x^2+1}$

$u'(x) = -\frac{1}{3}x^{-4/3} + e^{-x^2+1} \cdot -2x$

(e) [10 pts] $u(x) = \cos(\ln(2 + (x^2 - 1)^3))$

$u'(x) = -\sin\left[\ln(2+(x^2-1)^3)\right] \cdot \frac{1}{2+(x^2-1)^3} \cdot 3(x^2-1)^2 \cdot 2x$

(f) [10 pts] $v(x) = \frac{\tan(x^5 + 1)}{x + 1}$ $= \tan(x^5+1) \cdot (x+1)^{-1}$

$v'(x) = \tan(x^5+1) \cdot (-(x+1)^{-2}) + (x+1)^{-1} \cdot \sec^2(x^5+1) \cdot 5x^4$

or $v'(x) = \frac{(x+1) \cdot \sec^2(x^5+1) \cdot 5x^4 - \tan(x^5+1) \cdot 1}{(x+1)^2}$

2. Determine the anti-derivative represented by each of the following integrals.

(a) [10 pts] $\int x^3 - \sqrt{x}\, dx = \int (x^3 - x^{1/2})\, dx$

$= \frac{1}{4}x^4 - \frac{2}{3}x^{3/2} + C$

(b) [10 pts] $\int e^{2x}\, dx = \frac{1}{2}e^{2x} + C$

3. [5 pts] Evaluate the derivative

$$\frac{d}{dx}\int_0^x t \cdot \sec(5t)\, dt.$$

$$= x \sec(5x)$$

because of the Fundamental Theorem of Calculus

 Points deducted: ____ out of a possible 5 points

4. [10 pts] Determine the maximum and the minimum values of the function

$$f(x) = xe^{x-x^2}$$

where $0 \leq x \leq 3$.

$f'(x) = x \cdot e^{x-x^2}(1-2x) + e^{x-x^2} \cdot 1 = e^{x-x^2}(x-2x^2) + e^{x-x^2}$

$f'(x) = e^{x-x^2}(-2x^2+x+1)$

f' does not exist: none

$f'=0$: $-2x^2+x+1=0$

$(2x+1)(x+1)=0$

$x = \cancel{-1/2}$ $x=1$

not in interval

x	$f(x)$
0	0 ← abs min is 0
1	$1 \cdot e^0 = 1$ ← abs max is 1
3	$3e^{3-9} = 3e^{-6} \approx .007$

5. [10 pts] Find the equation of the tangent line to

$$x^2 - y^2 = y.$$

at the point $(x, y) = (-\sqrt{2}, 1)$.

$$2x - 2y\frac{dy}{dx} = \frac{dy}{dx}$$

$$2x = 2y\frac{dy}{dx} + \frac{dy}{dx}$$

$$2x = (2y+1)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x}{2y+1}$$

$$\left.\frac{dy}{dx}\right|_{(-\sqrt{2},1)} = \frac{-2\sqrt{2}}{2\cdot 1 + 1} = \frac{-2\sqrt{2}}{3}$$

$$y - y_1 = m(x - x_1)$$

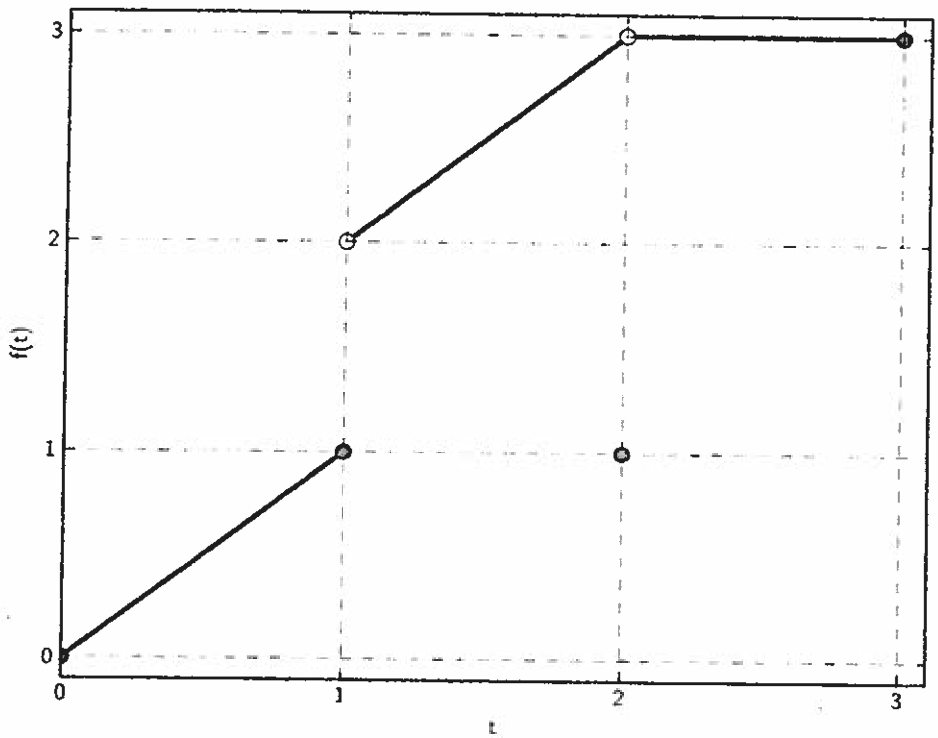
$$\boxed{y - 1 = \frac{-2\sqrt{2}}{3}(x + \sqrt{2})}$$

or $y = \frac{-2\sqrt{2}}{3}x - \frac{4}{3} + 1$

$$y = \frac{-2\sqrt{2}}{3}x - \frac{1}{3}$$

Points deducted: ____
out of a possible 10 points

6. A function, $f(t)$, is shown in the plot below. Determine the value of each of the following quantities. Briefly state the reason for your result in one or two **complete** sentences.



(a) [3 pts] $\lim_{t\to 1} f(t) =$ does not exist

The one-sided limits do not agree.

(b) [2 pts] $f(1) =$ 1

There is a solid dot at (1,1).

(c) [3 pts] $\lim_{t\to 2} f(t) =$ 3

The function values approach 3 as t approaches 2 from either side.

(d) [2 pts] $f(2) =$ 1

There is a solid dot at (2,1).

Points deducted: ____
out of a possible 10 points

7. [10 pts] Approximate the integral

$$\int_0^2 \cos(\pi x)\, dx$$

using a Riemann sum with three intervals. The intervals should be equal length and use a left hand sum. (You do not have to evaluate your result and can leave it as a sum, but it must be in a form that can be directly entered into a calculator.)

$\Delta x = \frac{2-0}{3} = \frac{2}{3}$ $\quad [\underset{*}{0}, \frac{2}{3}] [\underset{*}{\frac{2}{3}}, \frac{4}{3}] [\underset{*}{\frac{4}{3}}, 2]$

$f(0)\cdot\frac{2}{3} + f(\frac{2}{3})\cdot\frac{2}{3} + f(\frac{4}{3})\cdot\frac{2}{3}$

$= \cos(0)\cdot\frac{2}{3} + \cos(\frac{2\pi}{3})\cdot\frac{2}{3} + \cos(\frac{4\pi}{3})\cdot\frac{2}{3}$

$= 1\cdot\frac{2}{3} + \frac{-1}{2}\cdot\frac{2}{3} + \frac{-1}{2}\cdot\frac{2}{3}$

$= 0$

Points deducted: _____
out of a possible 10 points

8. [10 pts] Use the definition of the derivative to show that

$$\frac{d}{dx}\left(3x^2 - \frac{1}{x}\right) = 6x + \frac{1}{x^2}.$$

(hint: the quotient can be broken into two parts, one for each function.)

$$\frac{d}{dx}\left(3x^2 - \frac{1}{x}\right) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left(3(x+h)^2 - \frac{1}{x+h}\right) - \left(3x^2 - \frac{1}{x}\right)}{h}$$

$$= \lim_{h\to 0}\left(\frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} - \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \right)$$

$$= \lim_{h\to 0}\left(\frac{6xh + 3h^2}{h} - \frac{1}{h}\left(\frac{x}{(x+h)(x)} - \frac{(x+h)}{(x+h)(x)}\right)\right)$$

$$= \lim_{h\to 0}\left(\frac{h(6x+3h)}{h} - \frac{1}{h}\left(\frac{-h}{(x+h)(x)}\right)\right)$$

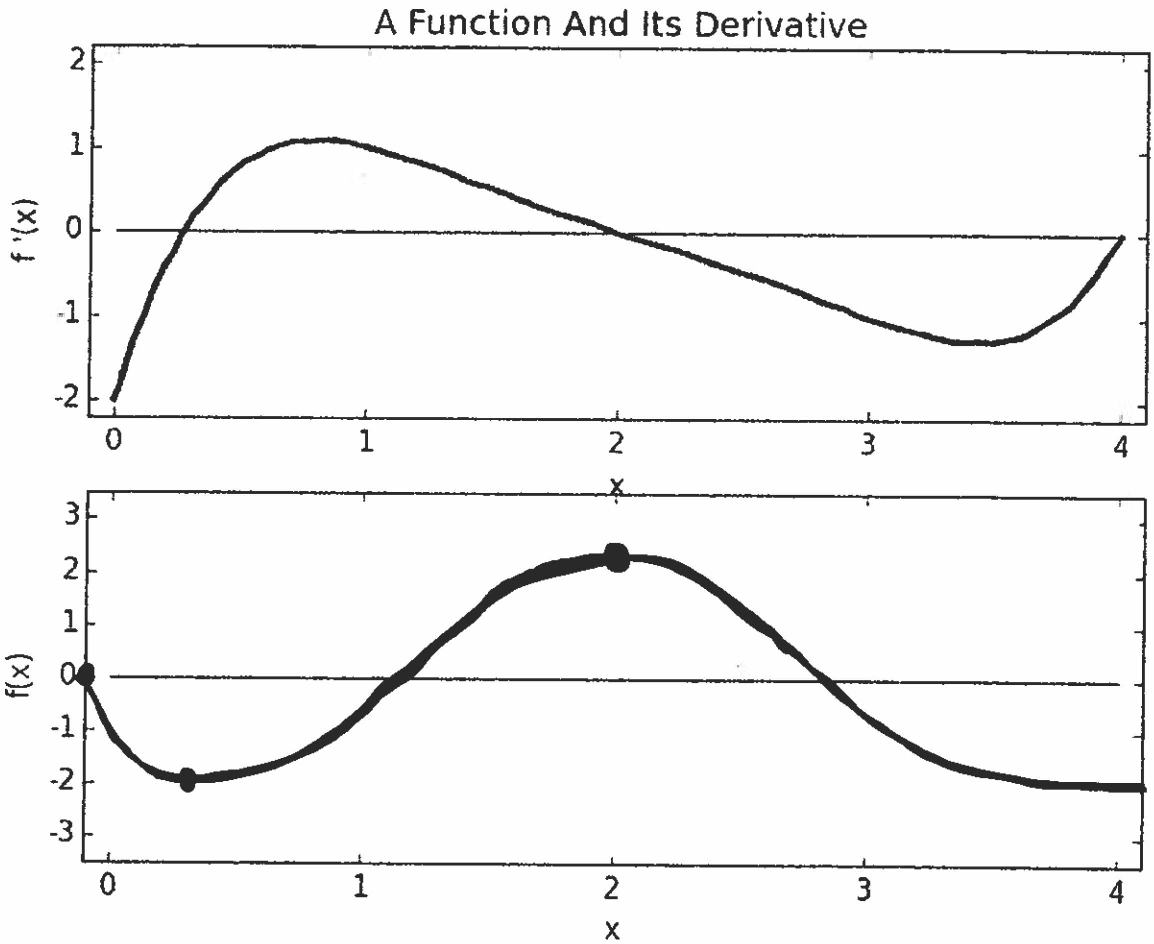
$$= \lim_{h\to 0}\left(6x + 3h + \frac{1}{(x+h)(x)}\right)$$

$$= 6x + \frac{1}{x^2}$$

Points deducted: ____
out of a possible 10 points

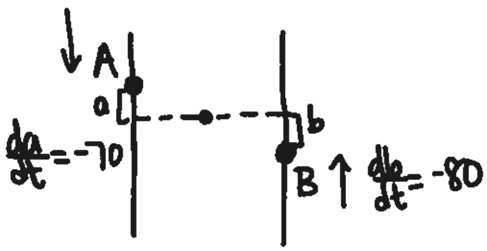
9. [10 pts] A sketch of the derivative of a function, $f'(x)$, is shown in the plot below. Make a sketch of the original function, $f(x)$, given that $f(0) = 0$.

(Please double check the labels on the plots.)

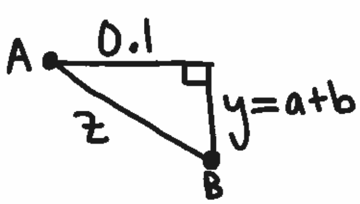


Points deducted: ____
out of a possible 10 points

10. [10 pts] A road has two lanes going north and south, and the lanes are separated by a distance of 0.1 miles. One car, traveling North, is traveling at a constant 80 miles per hour. Another car, traveling South is traveling at constant 70 miles per hour. What is the rate of change of the straight line distance between the two cars when they are approaching one another and the straight line distance between the cars is one mile? What is the rate of change of the straight line distance at the moment when they pass each other?



goal: $\frac{dz}{dt}$ when $z=1$ and when $z=0.1$



$$y^2+(0.1)^2=z^2$$
$$2y\frac{dy}{dt}=2z\frac{dz}{dt}$$
$$\frac{y}{z}\frac{dy}{dt}=\frac{dz}{dt}$$

$$y=a+b$$
$$\frac{dy}{dt}=\frac{da}{dt}+\frac{db}{dt}=-70+-80$$
$$=-150$$

when $z=1$: $y^2+(0.1)^2=1^2$
$$y^2=1-.01=.99$$
$$y=\sqrt{.99}$$

$\frac{dz}{dt}=\frac{\sqrt{.99}}{1}\cdot -150$ miles per hour

when $z=0.1$ (passing each other)

$\frac{dz}{dt}=0$ mph since z is decreasing before that time and increasing afterwards

11. [10 pts] Sketch a plot of the function

$$f(x) = \frac{2x}{x^2 - 4}.$$

Label your axes and indicate the values of all x-intercepts, y-intercepts, and asymptotes including for $x \to -\infty$ as well as $x \to \infty$.

x-intercept: $y=0$ when $x=0$ y-intercept: $f(0)=0$ so $y=0$

Horizontal asymptotes: $\lim\limits_{x\to\pm\infty} \frac{2x}{x^2-4} = \lim\limits_{x\to\pm\infty} \frac{2/x}{1-4/x^2} = 0$

so $y=0$ is a horizontal asymptote.

vertical: $x = \pm 2$

$\lim\limits_{x\to 2^-} \frac{2x}{(x+2)(x-2)} = -\infty$ $\left(\frac{\text{positive}}{\text{small } -\text{, going to } 0}\right)$; $\lim\limits_{x\to 2^+} \frac{2x}{(x+2)(x-2)} = \infty$ denominator is positive now

$\lim\limits_{x\to -2^-} \frac{2x}{(x+2)(x-2)} = -\infty$ $\left(\frac{\text{negative}}{(\text{neg, going to } 0)(\text{neg})}\right)$; $\lim\limits_{x\to -2^+} \frac{2x}{\underbrace{(x+2)}_{\text{This factor is positive now.}}(x-2)} = \infty$

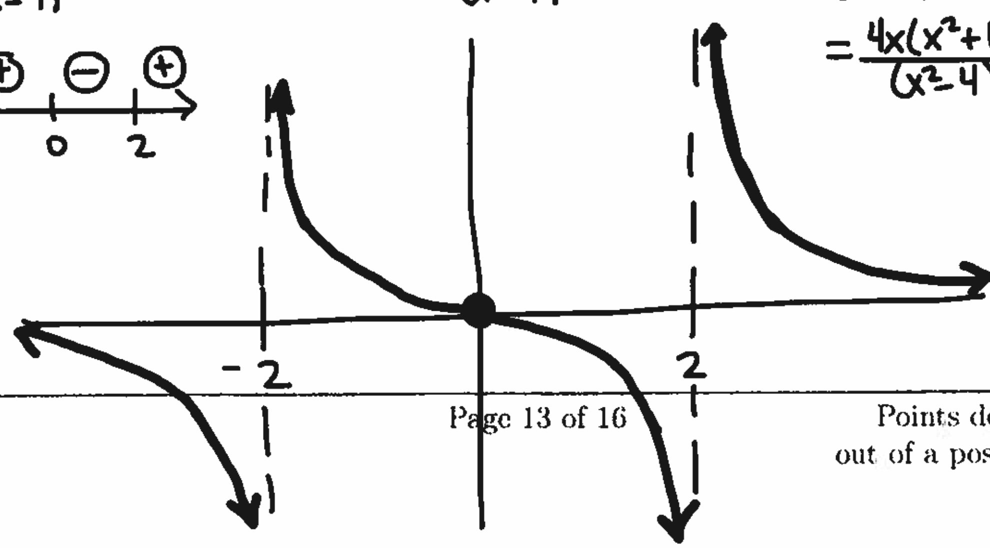
$f'(x) = \frac{(x^2-4)(2)-(2x)(2x)}{(x^2-4)^2} = \frac{-2x^2-8}{(x^2-4)^2} = \frac{-2(x^2+4)}{(x+2)^2(x-2)^2}$

⊖ −2 ⊖ 2 ⊖

$f''(x) = \frac{(x^2-4)^2(-4x)-(-2x^2-8)(2(x^2-4)2x)}{(x^2-4)^4} = \frac{(x^2-4)(-4x)+(2x^2+8)(4x)}{(x^2-4)^3} = \frac{-4x^3+16x+8x^3+32x}{(x^2-4)^3} = \frac{4x^3+48x}{(x^2-4)^3}$

$= \frac{4x(x^2+12)}{(x^2-4)^3}$

⊖ −2 ⊕ 0 ⊖ 2 ⊕



Points deducted: _____ out of a possible 10 points

12. A car starts from rest at a stop light. At the end of 10 seconds its position is 100 meters beyond the light. Three statements are given below. For each statement indicate if it must be true, must be false, or if it is not possible to determine indicate that you cannot tell from the given information. For each statement provide a complete, one sentence explanation for your reasoning.

(a) [3 pts] **True/False/Cannot Tell** Its final speed is 10 meters per second

We do not know the speed of the car; it may or may not be traveling 10 m/s at that time.

(b) [4 pts] **True/False/Cannot Tell** At some point in time its speed was 10 meters per second.

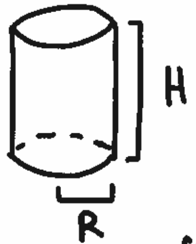
If $x(t)$ is the position of the car at time t, we know $\frac{x(10)-x(0)}{10-0} = \frac{100-0}{10} = 10$. By the Mean Value Theorem, $x'(c)=10$ for some c in $(0,10)$.

(c) [3 pts] **True/False/Cannot Tell** It did not move faster than 10 meters per second at any time.

Since the car started at rest ($x'(0)=0$), according to the Intermediate Value Theorem, the car was going slower than 10 m/s for some period of time. If the car traveled at most 10 m/s for the rest of the time, it couldn't have covered 100 meters in 10 seconds.

Points deducted: ____
out of a possible 10 points

13. [10 pts] A cistern for storing water will be constructed. Its shape is a right circular cylinder with radius R and height H. It must be able to hold 1000 m^3 of water. The cost of the materials is related to the surface area. It is \$8 per square meter for the sides and is \$10 per square meter for the bottom. (The top of the cistern is open) What dimensions for the cistern will minimize the cost of the materials?



goal: minimize cost

constraint equation: $\pi R^2 H = 1000$ or $H = \frac{1000}{\pi R^2}$

cost $= \left(\text{cost per unit}\right) \times \left(\#\text{ of units}\right)$

$$C = (10)(\pi R^2) + (8)(2\pi R H) = 10\pi R^2 + 16\pi R H$$

$$C = 10\pi R^2 + 16\pi R\left(\frac{1000}{\pi R^2}\right)$$

$$C = 10\pi R^2 + \frac{16000}{R}$$ domain: $(0, \infty)$

$$C' = 20\pi R - \frac{16000}{R^2} = \frac{20\pi R^3 - 16000}{R^2} = \frac{20(\pi R^3 - 800)}{R^2}$$

C' is defined on $(0,\infty)$ $\quad$ $C' = 0$: $\pi R^3 - 800 = 0$

$$R = \sqrt[3]{\frac{800}{\pi}} \approx 6.338$$

First derivative test:

	$\left(0, \sqrt[3]{\frac{800}{\pi}}\right)$	$\left(\sqrt[3]{\frac{800}{\pi}}, \infty\right)$
C'	$C'(1) = \frac{20(\pi - 800)}{1}$ $\ominus$	$C'(10) = \frac{20(1000\pi - 800)}{100}$ $\oplus$
C	decreasing	increasing

The cost function has a local minimum at $R = \sqrt[3]{\frac{800}{\pi}}$. Since there is only one critical number in $(0,\infty)$, this local minimum is also the absolute minimum. The lowest-cost cylinder has dimensions $R = \sqrt[3]{\frac{800}{\pi}}$ meters and

$$H = \frac{1000}{\pi\left(\sqrt[3]{\frac{800}{\pi}}\right)^2} \text{ meters.}$$

Points deducted: ____
out of a possible 10 points

Extra space for work. If you want us to consider the work on this page you must write your name and section number at the top of this page.