Topology Qualification Exam, Spring 2023

Instructions: You can assume homology groups and fundamental groups of a *point* and *wedges of spheres in all dimensions.* Everything else should be computed. All problems have equal weight.

1. Prove that a metric space is Hausdorff.

2. Let $\{X_i \mid i \in I\}$ be a collection of topological spaces indexed by an indexing set I. Let $X = \prod_{i\in I} X_i$ be the Cartesian product. Recall that there are two natural topologies one might put on X, the *box topology*, with basis equal to the set of sets of the form $\prod_{i\in I} U_i$ for all possible open $U_i \subset X_i$, and the *product topology*, with the same basis elements except that in each product $\prod_{i\in I} U_i$, all but finitely many U_i are required to equal the total space X_i. Give an example where X with the product topology is not homeomorphic to X with the box topology.

3. Describe a path-connected 3-sheeted covering space $p : \widetilde{X} \to X$ of $X = \mathbb{R}P^2 \vee S^1$. Make sure to describe both the space $\widetilde{X}$ and the map p. Let $x_0 \in X$ denote the point at which the wedge operation is performed to create $\mathbb{R}P^2 \vee S^1$. Given that $\pi_1(X, x_0) \cong (\mathbb{Z}/2\mathbb{Z}) * \mathbb{Z}$, fix some $\widetilde{x}_0 \in p^{-1}(x_0)$ and explicitly describe the subgroup $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) \subset \pi_1(X, x_0)$ in terms of the description of $\pi_1(X, x_0)$ as $(\mathbb{Z}/2\mathbb{Z}) * \mathbb{Z}$.

4. Explicitly describe a path-connected space X with basepoint $x_0 \in X$ such that $\pi_1(X, x_0) \cong \mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})$.

5. Consider a regular octagon P in the plane with opposite sides identified by a rigid translation of the plane. In other words, consider the equivalence relation $\sim$ on P where for two distinct points $p, q \in P$, $p \sim q$ if and only if p and q are on the boundary of P and there is a rigid translation of the plane taking one edge of P to an opposite edge and taking p to q. This produces an orientable surface $\Sigma = P/\sim$.

   (a) Calculate the genus of Σ.

   (b) Let $\rho : P \to P$ be rotation by π about the center point of P. Note that since $p \sim q$ implies $\rho(p) \sim \rho(q)$, ρ descends to a map $\rho : \Sigma \to \Sigma$. (You do not need to prove that fact.) How many fixed points does $\rho : \Sigma \to \Sigma$ have?

   (c) We claim that Σ/ρ is a surface (do not prove this); what is the genus of Σ/ρ?

6. Decompose $S^1 \times S^n$ as $(S^1 \times S^n_+) \cup (S^1 \times S^n_-)$, where

$$S^n_\pm = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1 \text{ and } \pm x_0 > -1/2\}.$$

   Use the Mayer-Vietoris sequence for this decomposition to show that $H_k(S^1 \times S^n) \cong H_{k-1}(S^1 \times S^{n-1})$ for all $k \geq 3$ and for all $n \geq 1$. (This is also true for other values of k and n but this is the easiest case to prove.)

7. Let B be the closed unit ball in $\mathbb{R}^3$, let S be the circle of radius $1/2$ centered at the origin in the xy plane in $\mathbb{R}^3$, and let $P = (0, 0, 0)$. Compute the homology of $X = B \setminus (S \cup \{P\})$.

8. Using cylindrical coordinates (r, θ, z) on S^2, consider the function $f_n : S^2 \to S^2$ given by $f_n(r, \theta, z) = (r, n\theta, z)$ for some $n \in \mathbb{Z}$. Use cellular homology to compute all homology groups of the space X obtained by gluing B^3 to S^2 using the map f_n (thought of as a map from the boundary of B^3 to S^2).