Topology Qualifying Examination
January, 2016

Instructions: Work all problems. Give clear explanations and complete proofs.

(1) Prove that if $X$ and $Y$ are connected topological spaces then $X \times Y$ is connected.

(2) In each part of this problem $X$ is a compact topological space. Give a proof or a counterexample for each statement.

(a) If $\{F_n\}_{n=1}^{\infty}$ is a sequence of nonempty closed subsets of $X$ such that $F_{n+1} \subset F_n$ for all $n$ then $\cap_{n=1}^{\infty} F_n$ is nonempty.

(b) If $\{O_n\}_{n=1}^{\infty}$ is a sequence of nonempty open subsets of $X$ such that $O_{n+1} \subset O_n$ for all $n$ then $\cap_{n=1}^{\infty} O_n$ is nonempty.

(3) Recall that the *suspension* of a topological space $X$, denoted $SX$, is the quotient space formed from $X \times [-1, 1]$ by identifying $(x, 1)$ with $(y, 1)$ for all $x, y \in X$, and also identifying $(x, -1)$ with $(y, -1)$ for all $x, y \in X$.

(a) Show that $SX$ is the union of two contractible subspaces.

(b) Prove that if $X$ is path-connected then $\pi_1(SX) = \{0\}$.

(c) For all $n \geq 1$, prove that $H_n(X) \cong H_{n+1}(SX)$.

(4) Prove that the free group on two generators contains a subgroup isomorphic to the free group on five generators by constructing an appropriate covering space of $S^1 \vee S^1$.

(5) Let $X$ be the quotient space

$$X = \frac{S^1 \times [0, 1]}{(e^{i\theta}, 1) \sim (e^{3i\theta}, 0) \text{ for all } e^{i\theta} \in S^1}$$

Show how to give $X$ the structure of a cell complex, and use this to compute the homology (in all degrees) of $X$.

(6) Give a list without repetitions of all compact surfaces (orientable or non-orientable and with or without boundary) that have Euler characteristic negative one. Explain why there are no repetitions on your list.

(7) Use the Brouwer fixed point theorem to show that an $n \times n$ matrix with non-negative entries has a real eigenvalue.

(8) Given an example, with explanation, of a closed curve in a surface which is not null homotopic but is null homologous.