TOPOLOGY QUALIFYING EXAM, FALL 2017

1. Let $f : X \to Y$ be a continuous function between topological spaces. Let A be a subset of X and let $f(A)$ be its image in Y. One of the following statements is true and one is false. Decide which is which, prove the true statement, and provide a counterexample to the false statement:

(1) If A is closed then $f(A)$ is closed.
(2) If A is compact then $f(A)$ is compact.

2. Let $\{Y_n\}_{n=1}^{\infty}$ be a sequence of connected subsets of a topological space X. Suppose that $Y_n \cap Y_{n+1}$ is nonempty for all n. Show that $\bigcup_n Y_n$ is connected.

3. A metric space X is said to be **totally bounded** if for every $\epsilon > 0$ there exists a finite cover of X by open balls of radius ϵ.

a) Show: a metric space X is totally bounded iff every sequence in X has a Cauchy subsequence.

b) Exhibit a complete metric space X and a closed subset A of X that is bounded but not totally bounded. (You are not required to prove that your example satisfies the stated properties.)

4. (a) Prove that every continuous map $f : S^2 \to S^1 \times S^1$ is homotopic to a constant map.

(b) Prove that the quotient map $f : S^2 \to \mathbb{R}P^2$ is **not** homotopic to a constant map. (Here we regard $\mathbb{R}P^2$ as the quotient of S^2 by the antipodal map.)

5. Describe, as explicitly as you can, two different (non-homeomorphic) connected two-sheeted covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^3$, and prove that they are not homeomorphic.

6. Let M be a compact orientable surface of genus 2 without boundary. Give an example of a pair of loops $\gamma_0, \gamma_1 : S^1 \to M$ with $\gamma_0(1) = \gamma_1(1)$ such that there is a continuous map $\Gamma : [0,1] \times S^1 \to M$ such that $\Gamma(0,t) = \gamma_0(t), \Gamma(1,t) = \gamma_1(t)$ for all $t \in S^1$, but such that there is **no** such map Γ with the additional property that $\Gamma(s,1) = \gamma_0(1)$ for all $s \in [0,1]$. (You are not required to prove that your example satisfies the stated property.)

7. Compute, by any means available, the fundamental group and all the homology groups of the space obtained by gluing one copy A of S^2 to another copy B of S^2 via a 2-sheeted covering space map from the equator of A onto the equator of B.

8. Let $A \subset X$. Prove that the relative homology group $H_0(X, A)$ is trivial if and only if A intersects every path component of X.

9. Prove that, if $n \neq 2$, then $\mathbb{R}^2$ and $\mathbb{R}^n$ are not homeomorphic.