Topology Qualifying Exam
January 5, 2011

You have 3 hours. The grading committee will place a particular emphasis on finished and rigorous solutions for the problems, as opposed to partial attempts. Therefore, it is better to complete a smaller number of questions in the time allotted than to submit partially completed work for all the problems. Please carefully justify all of your answers.

1. (a) (5 points) What does it mean to say that $p : Y \to X$ is a *covering map*?
   (b) (5 points) Let $p : Y \to X$ be a covering map. Prove that, for any $y \in Y$, the homomorphism
   $$p_* : \pi_1(Y, y) \to \pi_1(X, p(y))$$
   is injective.
   (c) (5 points) Let $p : Y \to X$ be a covering map with Y and X path-connected. Suppose that p_* is an isomorphism. Prove that p is a homeomorphism.
2. (10 points) Let X be the topological space obtained by identifying three distinct points on the torus $S^1 \times S^1$. Calculate the fundamental group of X.
3. (10 points) Let X be a topological space obtained by attaching a 2-cell to $\mathbb{R}P^2$ via some map $f : S^1 \to \mathbb{R}P^2$. What are the possibilities for the homology groups $H_*(X; \mathbb{Z})$?
4. (10 points) Show that $\mathbb{R}P^2 \vee S^1$ is not homotopy equivalent to a compact surface (possibly with boundary).
5. (a) (5 points) State the Lefschetz Fixed Point Theorem for a finite simplicial complex X.
   (b) (10 points) Use degree theory to prove this Theorem in the case that $X = S^n$.
6. (10 points) Show that a compact subset of a Hausdorff space is closed.
7. (10 points) A topological space is *totally disconnected* if its only connected subsets are one-point sets. Is it true that if X has the discrete topology, it is totally disconnected? Is the converse true? Justify your answers.
8. Recall that a topological space X is said to be *regular* if for every point $p \in X$ and closed subset $F \subset X$ not containing p, there exists disjoint open sets U, V with $p \in U$ and $F \subset V$. Let X be a regular space that has a countable basis for its topology, and let U be an open subset of X.

   (a) (10 points) Show that U is a countable union of closed subsets of X.
   (b) (10 points) Show that there is a continuous function $f : X \to [0, 1]$ such that $f(x) > 0$ for $x \in U$ and $f(x) = 0$ for $x \notin U$.