Ph.D. Comprehensive Examination on Algebra

Spring 2003

You have three hours to complete this exam. Please write your solutions in a clear and concise fashion.

Complete each of the following problems

1. Let F be a field and $p(x)$ be a polynomial in $F[x]$. Show that the ideal $(p(x))$ in $F[x]$ is maximal if and only if $p(x)$ is irreducible over F.

2. State the three Sylow theorems. Prove that there are no simple groups of order 182.

3. Compute the Galois group of $x^4 - 2$ over $\mathbb{Q}$.

4. Let G be a finite group and H be a subgroup of G. Prove that G is solvable if and only if H is solvable and G/H is solvable.

5. Prove the Cayley Hamilton Theorem: every square matrix satisfies its characteristic polynomial.

6. Let A and B be two $n \times n$ matrices with the property that $AB = BA$. Suppose that A and B are diagonalizable. Prove that A and B are simultaneously diagonalizable.

7. Let R be a ring with the following commutative diagram of R-modules with each row representing a short exact sequence of R-modules and all maps being R-module homomorphisms.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \end{array}$$

   Prove that if α and γ are injective maps then β is injective.

8. A commutative ring R is Noetherian if and only if it satisfies the ascending chain condition on ideals. Prove that R is Noetherian if and only if every ideal of R is finitely generated.