Numerical Analysis : MAT 603-605

MAMS Comprehensive Exam
Spring 1996

Name: Score: $\overline{}$ 100

Instructions: Please work any 5 and only 5 of the following 6 problems. Each of the problems is worth 20 points. Please start each problem on a separate sheet of paper and write on only one side of each page.

1. Kepler's equation
$$m = x - E\sin x,$$
where m and E are given and x is sought, plays a considerable role in dynamical astronomy. Use one-point (linear or fixed point) iteration to iteratively compute the solutuion to Kepler's equation correct to 2-decimal places if $m = 0.8$ and $E = 0.2$. Justify your accuracy claim.
2. Let $f \in C^2[x_0, x_1]$ and let $p(x)$ be a polynomial of degree ≤ 1 that interpolates to $f(x)$ at the two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$. Use the classical theorem on polynomial interpolation error to prove that
$$\max_{x_0 \leq x \leq x_1} |f(x) - p(x)| \leq \frac{1}{8}(x_1 - x_0)^2 \max_{x_0 \leq x \leq x_1} |f''(x)|.$$
3. Use the composite trapezoidal rule and a calculator to approximate
$$\int_0^1 \frac{\sin x}{x} dx,$$
if the step size $h = .25$.
4. Determine a, b, α and β so that the one-step method
$$\begin{aligned} y_{n+1} &= y_n + ak_1 + bk_2 \\ k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \alpha h, y_n + \beta k_1) \end{aligned}$$
has local truncation error $O(h^3)$.
5. Let A be an $n \times n$ diagonal matrix with diagonal entries all equal to 10^{-1},
$$A = \text{diag}(10^{-1}, 10^{-1}, \ldots, 10^{-1}).$$
   a) Compute the determinant of A, $\det A$.
   b) Compute the condition number $K(A)$ of A relative to the 1-matrix norm $\|\cdot\|_1$.
   c) Some people believe that the linear system $Ax = b$ is ill-conditioned if $|\det A| << 1$. In view of (a) and (b) above is this belief justified? Explain.
6. Use the method of undetermined coefficients to derive the 2-point Gauss-Legendre quadrature rule.