Algebra Preliminary Exam, May 1995

1. Prove that every group of order 80 has a proper, normal subgroup.

2. Suppose the alternating group A_5 operates (i.e., acts) on a set S. Prove that no element of S has an orbit of order 2.

3. Let R be a commutative local ring with identity. Let m be the maximal ideal of R and let M be a finitely generated R-module. Prove that if M/mM is an n-dimensional vector space over the field R/m, then M can be generated by n elements.

4. Let $\xi_{13} = e^{\frac{2\pi}{13}} \in \mathbb{C}$ (so ξ_{13} is a primitive 13th root of unity). Let $\omega = \xi_{13} + \xi_{13}^5 + \xi_{13}^8 + \xi_{13}^{12}$.

   (a) Find the subgroup H of the Galois group $\mathrm{Gal}(\mathbb{Q}(\xi_{13})/\mathbb{Q})$ corresponding to $\mathbb{Q}(\omega)$ in the Galois correspondence and prove your answer carefully.

   (b) Explain why $\mathbb{Q}(\omega)$ is a Galois extension of $\mathbb{Q}$ and describe $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ (include the order of $\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q})$ in your description).

5. Let $f(x), g(x) \in \mathbb{Q}[x]$ be irreducible polynomials (over $\mathbb{Q}$) and let $\alpha, \beta \in \mathbb{C}$ such that $f(\alpha) = 0$, $g(\beta) = 0$. Prove that if $g(x)$ is irreducible over $\mathbb{Q}(\alpha)$, then $f(x)$ is irreducible over $\mathbb{Q}(\beta)$.

6. Let A, B be $n \times n$ symmetric matrices with entries in the real numbers. Prove that if $AB = BA$ then there is an invertible matrix $n \times n$ matrix M such that MAM^{-1} and MBM^{-1} are both diagonal matrices.

7. Describe, in as much detail as possible, all the finitely generated, projective $\mathbb{Z}$-modules and prove your answer.

8. Let p be a prime number and let m, n be positive integers. Let $\mathbb{F}_{p^m}$ denote the finite field of order p^m. Prove that the polynomial $x^n - 1$ has exactly n distinct roots in $\mathbb{F}_{p^m}$ if and only if $p^m \equiv 1 \pmod{n}$. (Hint: consider the group $\mathbb{F}_{p^m}^\times$ of nonzero elements of $\mathbb{F}_{p^m}$ under multiplication.)

9. For each of the following, either give an example or state that is none.

   (a) A PID (principal ideal domain) that is not a UFD (unique factorization domain).

   (b) A UFD that is not a PID.

   (c) A commutative ring R, an R-module M and an exact sequence of R-modules $0 \to A \to B$ such that $0 \to A \otimes M \to B \otimes M$ is not exact.

   (d) A non-abelian group of order p^2 where p is a prime number.