Topology Preliminary Exam
1:00–4:00 pm, Friday, September 12, 1997

1. (a) If A is a closed bounded subset of the metric space X, is A compact? Give a proof or a counterexample.

(b) If Y is a quotient space of the metric space X, is Y Hausdorff? Give a proof or a counterexample.

2. (a) State the Tietze extension theorem.

(b) Show that a connected normal space having more than one point is uncountable.

3. Prove that the metric space (X, d) is complete if and only if for every nested sequence $A_1 \supseteq A_2 \supseteq \cdots$ of nonempty closed subsets of X such that diameter$(A_n) \to 0$, we have $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

4. Two covering spaces $p : Y \to X$ and $q : Z \to X$ are *equivalent* if there is a homeomorphism $h : Y \to Z$ such that $p = q \circ h$. Find three connected two-sheeted covering spaces of the torus $X = S^1 \times S^1$ such that no two of the three are equivalent. Prove that there are exactly three equivalence classes of connected two-sheeted covering spaces of the torus. Give precise statements of all general theorems about covering spaces that you use. (You do not have to prove these theorems.)

5. The *dunce cap* is the space X obtained from the unit disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$ by making the following identifications of points on the boundary of D. For $0 \leq \theta \leq 2\pi/3$,

$$(\cos\theta, \sin\theta) \sim (\cos(\theta + 2\pi/3), \sin(\theta + 2\pi/3)),$$
$$(\cos\theta, \sin\theta) \sim (\cos(-\theta), \sin(-\theta)).$$

Compute the fundamental group of X.

6. Suppose that the CW complex X has one 0-cell, one 1-cell, one 2-cell, one 3-cell, and no cells of dimension greater than 3. What can you say about the homology of X? In other words, determine all possible sequences of groups $(G_0, G_1, G_2, \ldots)$ such that $G_i = H_i(X)$ for $i = 0, 1, 2, \ldots$, and X is such a CW complex.

7. Let $X = \mathbb{C}P^2$, the complex projective plane. Prove that if $f : X \to X$ is a continuous map homotopic to the identity map, then f has a fixed point.

8. (a) State the Mayer-Vietoris theorem for singular homology.

(b) Use it to prove by induction on n that

$$\widetilde{H}_i(S^n) \cong \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n \end{cases}$$

for $n \geq 0$, where $\widetilde{H}_i$ denotes reduced singular homology, and S^n is the n-sphere.