Analysis Qualifying Exam: Real Analysis
August 2004

Give clear reasoning. State clearly which theorem you are using. Using the back page for rough work. Cross out the parts you do not want to be graded. Read through all the problems, do them in any order, the one you feel most confident about first. They are not in the order of difficulty. You should not cite anything else: examples, exercises, or problems.

Problem #	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total	50	

Committee Recommendation

Grader's Remark

1. Let $x_0 = a$ and $x_1 = b$, and continue the sequence by letting each new term be the average of the preceding two:

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}, \quad n \geq 2$$

(a) Prove that $\{x_n\}$ is a Cauchy sequence.

(b) Find the limit in terms of a and b.

2. Evaluate the integral

$$\phi(x) = \int_0^\infty \frac{e^{-t}(1 - \cos xt)}{t} dt.$$

Hint: you may need the formula $\int e^{au} \sin bu du = \frac{e^{au}}{a^2 + b^2}(a \sin bu - b \cos bu) + C.$

3. If $f \in L^+$, i.e., $f \geq 0$ and $\int f(x)d\mu < \infty$. Show that for any $\epsilon > 0$, there exists a measurable set E so $\mu(E) < \infty$ and $\int_E f(x)d\mu > \int f(x)d\mu - \epsilon$.

4. Let $f \in L^1(\mathbb{R})$ and $F(t) = \int_{\mathbb{R}} f(x)e^{itx}dx$. Prove:

(a) The function $F(t)$ is continuous on $\mathbb{R}$.

(b) $\lim_{t \to \infty} F(t) = 0$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. If for each $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exist $F \in \mathcal{M}$ with $F \subset E$ and $0 < \mu(F) < \infty$, then μ is called *semifinite.* Prove that if μ is semifinite measure, $E \in \mathcal{M}$ and $\mu(E) = \infty$, then for any positive real number C, there exists $F \subset E$ with $C < \mu(F) < \infty$.
Hint: this is equivalent to $\sup\{\mu(F) : \ F \in \mathcal{M}, \ F \subset E\} = \infty$.