Real Analysis Qualifying Examination

August 2011

There are five problems, each worth 20 points. Give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

1. Let f be a differentiable function on $[a,b]$. We say that f is *uniformly differentiable* on $[a,b]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that
$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon$$
whenever $|x - y| < \delta$ with $x, y \in [a,b]$.
   (a) Prove that f is uniformly differentiable on $[a,b]$ <u>if and only if</u> f' is continuous on $[a,b]$.
   (b) Give an example of a function that is differentiable on $[a,b]$ but fails to be uniformly differentiable on $[a,b]$ (no proofs required).

2. (a) Let $f : [0,1] \to \mathbb{R}$. Give a definition of what it means to say that f is a Lebesgue measurable function.
   (b) Let $\{f_k\}$ be a sequence of finite-valued measurable functions on [0,1]. Prove that
$$\limsup_{k\to\infty} f_k$$
is a measurable function.

3. Let $\{f_k\}$ be a sequence of Lebesgue integrable functions on $\mathbb{R}$. Recall that $\{f_k\}$ is said to *converge in measure* to 0 if for every $\varepsilon > 0$,
$$\lim_{k\to\infty} m(\{x \in \mathbb{R} : |f_k(x)| \geq \varepsilon\}) = 0,$$
where m stands for Lebesgue measure on $\mathbb{R}$.
   (a) Give an example to illustrate the fact that $f_k \to 0$ in measure does <u>not</u> necessarily imply $f_k \to 0$ in L^1.
   (b) Prove that if we make the additional assumption that there exists an integrable function g such that $|f_k| \leq g$ for all k, then $f_k \to 0$ in measure does imply that $f_k \to 0$ in L^1.

4. Let $\varphi \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$ and $\varphi_t(x) := t^{-n}\varphi(t^{-1}x)$. Prove that if $f \in L^1(\mathbb{R}^n)$, then
$$\lim_{t\to 0} \int_{\mathbb{R}^n} |f * \varphi_t(x) - f(x)|\,dx = 0,$$
where
$$f * \varphi_t(x) = \int_{\mathbb{R}^n} f(x-y)\varphi_t(y)\,dy$$
denotes the convolution of f with φ_t.

5. For each $1 \leq p \leq \infty$, define $\Lambda_p : L^p([0,1]) \to \mathbb{R}$ by
$$\Lambda_p(f) = \int_0^1 x^2 f(x)\,dx.$$
Explain why Λ_p is a bounded linear functional and compute its norm (in terms of p).