Topology Qualifying Exam, Fall 2002

1. The "line with two origins" is $X = (\mathbb{R} - \{0\}) \cup \{0^+, 0^-\}$ with topology defined to consist of open sets in $\mathbb{R} - \{0\}$ and of sets of the form $(U \setminus \{0\}) \cup \{0^+\} = U^+$ and $(U \setminus \{0\}) \cup \{0^-\} = U^-$ for $U \subset \mathbb{R}$ open and containing 0. Prove that this is indeed a topology, and that it is not metrizable.

2. Let $\{X_\alpha : \alpha \in \mathcal{A}\}$ be a family of sets such that the intersection of any two of them is nonempty. Prove that if X_α are connected subspaces of a space X, then their union $\bigcup_\alpha X_\alpha$ is a connected subspace of X.

3. Prove that a product of two compact spaces is compact.

4. Use van Kampen's theorem to compute the fundamental group of a compact orientable surface of genus 2.

5. Let X be the union of a torus $T = S^1 \times S^1$ and a disc D^2, attached via the map $\phi : S^1 = \partial D^2 \longrightarrow T$, $\phi(e^{i\theta}) = (e^{i\theta}, 1)$. Compute the homology $H_*(X)$ and the fundamental group $\pi_1(X)$. What is the universal cover $\tilde{X}$ of X? What is the action of $\pi_1(X)$ on $\tilde{X}$?

6. a.) Let Σ_2 be the orientable surface of genus 2. Suppose $\Sigma_g \xrightarrow{p} \Sigma_2$ is a 5-fold cover. What is g?

   b.) If Σ is a genus 2 surface with the interior of a single disc removed and $\tilde{\Sigma}$ is a 3-fold cover of Σ, what can $\tilde{\Sigma}$ be?

7. Prove that any map $\mathbb{R}P^2 \longrightarrow S^1 \times S^1$ is null homotopic. Prove that there exists a map $S^1 \times S^1 \longrightarrow \mathbb{R}P^2$ which is not null-homotopic.

8. Show that the space of matrices $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d, \in \mathbb{R}, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} > 0 \right\}$ is not contractible.

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