Algebra Preliminary Exam

Wednesday September 18, 1991

Questions 1-8 are worth 10 points. Question 9 is worth 20 points.

1. How many isomorphism classes of order 21 are there? Give a presentation for a group in each each isomorphism class. How do you know that you have a complete list?

2. Prove there is no simple group of order 80.

3. Let F be a finite field of order q. Prove that in $F[x]$, $x^{q^n} - x = \prod g(x)$, where the product is taken over all monic irreducible polynomials in $F[x]$ of degrees dividing n.

4. Suppose p is prime and f is an irreducible polynomial of degree p over $\mathbb{Q}$ which has exactly two non-real roots in $\mathbb{C}$. Prove that the Galois group of f is isomorphic to the symmetric group S_p.

5. Suppose that R is an infinite commutative ring with finitely many units. Prove that R has infinitely many prime ideals.

6. Prove that an element in a commutative ring R is nilpotent if and only if it is contained in every prime ideal of R.

7. Prove that $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$ where $m, n \in \mathbb{Z}$ and $d = \gcd(m, n)$.

8. Let A be an irreducible module over a ring R. Prove that $End_R(A)$ is a division ring.

9. For each of the following fields F with extension field E, determine the Galois group $Gal(E/F)$ and the field of invariants $Inv(Gal(E/F))$. What is the degree of E/F in each case?
   (a) $F = \mathbb{Q}$, $E = \mathbb{Q}(\sqrt{5}, \sqrt{7})$
   (b) $F = \mathbb{Q}$, $E = \mathbb{Q}(\sqrt[3]{2})$
   (c) $F = \mathbb{Z}/p\mathbb{Z}(t)$, the field of rational functions in an indeterminant t, with coefficients in $\mathbb{Z}/p\mathbb{Z}$ and $E = F(u)$, where u is a root of the polynomial $x^p - t$.