Analysis Qualifying Exam: Complex Analysis

Spring 2006

Show work and carefully justify/prove your assertions.

1. (1) Let $a > 0$. Evaluate $\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx$ using complex contour integral. Justify all steps.

   (2) Let $a < 0$. Based on (1), give a formula for $\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx$ without going into the steps as in (1) and briefly explain why the formula is correct.

2. Prove that if $f(z)$ is an analytic function in the complex plane $\mathbb{C}$ such that its real part $Re(f(z))$ is a polynomial in x, y, then $f(z)$ is a polynomial in z, i.e.,

$$f(z) = c_0 + c_1 z + \dots + c_m z^m$$

   for some complex constants $c_0, c_1, \dots, c_m$, where $z = x + iy$.

3. Give two different proofs of the fundamental theorem of algebra using methods in complex analysis (other methods will not count).

4. (a) Interpret what the following assertion means: "The series

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

   defines an analytic function in $Re(z) > 1$."

   (b) Prove the assertion in (a).

5. Let $a_n(z)$ be a sequence of analytic functions on the unit disk $D : |z| < 1$ such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed subsets of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed subsets of D.