Complex Analysis Qualifying Exam, Fall 2015

The problems count equally. Give clear reasoning and state clearly which theorems you are using. $\mathbb{D}$ denotes the open unit disk in the complex plane $\mathbb{C}$.

1. Evaluate the integral $\displaystyle\int_0^\infty \frac{\cos x}{1+x^4}\,dx$.

2. (a) Let $\{f_n\}$ be a sequence of analytic functions on a region Ω in $\mathbb{C}$ and assume that for some function g on Ω, $f_n \to g$ uniformly on compact subsets of Ω. Show that $f_n' \to g'$ uniformly on compact subsets of Ω and that g is analytic on Ω.

   (b) Explain what part (a) has to do with differentiation of power series.

3. Let U be the upper half plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ with the slit $\{i\,y \mid 0 < y \leq 1\}$ removed. Find a conformal equivalence from U to $\mathbb{D}$.

4. Expand the following functions into Laurent series in the indicated regions:

   (a) $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}, \quad 2 < |z| < 3.$

   (b) $g(z) = \sin\left(\dfrac{z}{1-z}\right), \quad 0 < |z-1| < \infty.$

5. Let f be entire and suppose that $\lim_{z\to\infty} f(z) = \infty$. Show that f is a polynomial.

6. Find the number of roots of $z^4 - 6z + 3 = 0$ in $|z| < 1$, $1 < |z| < 2$, and $|z| > 2$ respectively.

7. Let $f(z)$ be an analytic on $\mathbb{D}$, with $\operatorname{Re}(f(z)) > 0$, $f(0) = a > 0$. Show that

$$\left|\frac{f(z)-a}{f(z)+a}\right| \leq |z|, \quad |f'(0)| \leq 2a.$$