Complex Analysis Qualifying Exam 2019 Fall
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1. Show that $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}dx = \frac{\pi}{n\sin\frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

2. Prove that the distinct complex numbers z_1, z_2 and z_3 are the vertices of an equilateral triangle if and only if
$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

3. Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z\to\infty} f(z) = A$. Show that
$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

4. Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

5. Find a conformal map from $D = \{z : \ |z| < 1,\ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : \ |z| < 1\}$.

6. A holomorphic mapping $f : U \to V$ is a local bijection on U if for every $z \in U$ there exists an open disc $D \subset U$ centered at z so that $f : D \to f(D)$ is a bijection. Prove that a holomorphic map $f : U \to V$ is a local bijection if and only if $f'(z) \neq 0$ for all $z \in U$.