Algebra, Fall 2016

Problem 1 score (out of 10) _____

Problem 2 score (out of 10) _____

Problem 3 score (out of 10) _____

Problem 4 score (out of 10) _____

Problem 5 score (out of 10) _____

Problem 6 score (out of 10) _____

Problem 7 score (out of 10) _____

TOTAL SCORE (out of 70) _____

Problems

(1) (10 points) Let G be a finite group and let $s, t \in G$ be two distinct elements of order 2. Show that the subgroup of G generated by s and t is a dihedral group. Recall that the dihedral groups of order $2m$ are of the form

$$D_{2m} = < \sigma, \tau \mid \sigma^m = 1 = \tau^2, \tau\sigma = \sigma^{-1}\tau >,$$

for some $m \geq 2$.

(2) (10 points) Let A and B be two $n \times n$ matrices with the property that $A \cdot B = B \cdot A$. Suppose that A and B are diagonalizable. Prove that A and B are simultaneously diagonalizable.

(3) (10 Points) How many groups are there up to isomorphism of order pq, where $p < q$ are prime integers?

(4) (10 points) Set $f(x) = x^3 - 5 \in \mathbb{Q}[x]$.

(a) Find the splitting field K of $f(x)$ over $\mathbb{Q}$.
(b) Find the Galois group G of K over $\mathbb{Q}$.
(c) Exhibit explicitly the correspondence between subgroups of G and intermediate fields between $\mathbb{Q}$ and K.

(5) (10 points) How many monic irreducible polynomials over $\mathbb{F}_p$ of prime degree ℓ are there? Justify your answer.

(6) (10 points) Let R be a ring and $f : M \to N$ and $g : N \to M$ be R-module homomorphisms such that $g \circ f = \mathrm{id}_M$. Show that $N \cong \operatorname{Im} f \oplus \operatorname{Ker} g$.

(7) (a) (1 points) Define solvable for a group G.
(b) (9 points) Show that every group G of order 36 is solvable. *Hint:* You can use that S_4 is solvable.