ALGEBRA QUALIFYING EXAM, FALL 2019

Instructions: Complete all 8 problems. In multi-part problems, you may assume the result of any part (even if you have not been able to do it) in working on subsequent parts.

(1) Let G be a finite group with n distinct conjugacy classes . Let $g_1 \cdots g_n$ be representatives of the conjugacy classes of G. Prove that if $g_i g_j = g_j g_i$ for all i, j then G is abelian.

(2) Let G be a group of order 105 and let P, Q, R be Sylow $3, 5, 7$ subgroups respectively.
  (a) Prove that at least one of Q and R is normal in G.
  (b) Prove that G has a cyclic subgroup of order 35.
  (c) Prove that both Q and R are normal in G.
  (d) Prove that if P is normal in G then G is cyclic.

(3) Let R be a ring with the property that for every $a \in R, a^2 = a$
  (a) Prove that R has characteristic 2.
  (b) Prove that R is commutative.

(4) Let F be a finite field with q elements. Let n be a positive integer relatively prime to q and let ω be a primitive nth root of unity in an extension field of F. Let $E = F[\omega]$ and let $k = [E : F]$.
  (a) Prove that n divides $q^k - 1$.
  (b) Let m be the order of q in $\mathbb{Z}/n\mathbb{Z}$. Prove that m divides k.
  (c) Prove that $m = k$.

(5) Let R be a ring and M an R-module. Recall that the set of torsion elements in M is defined by $\mathrm{Tor}(m) = \{m \in M \mid \exists r \in R, r \neq 0, rm = 0\}$.
  (a) Prove that if R is an integral domain, then $\mathrm{Tor}(M)$ is a submodule of M.
  (b) Give an example where $\mathrm{Tor}(M)$ is not a submodule of M.
  (c) If R has 0-divisors, prove that every non-zero R-module has non-zero torsion elements.

(6) Let R be a commutative ring with multiplicative identity. Assume Zorn's Lemma.
  (a) Show that
$$N = \{r \in R \,\big|\, r^n = 0 \text{ for some } n > 0\}$$
  is an ideal which is contained in any prime ideal.
  (b) Let r be an element of R not in N. Let S be the collection of all proper ideals of R not containing any positive power of r. Use Zorn's Lemma to prove that there is a prime ideal in S.
  (c) Suppose that R has exactly one prime ideal P. Prove that every element r of R is either nilpotent or a unit.

(7) Let ζ_n denote a primitive nth root of $1 \in \mathbb{Q}$. You may assume the roots of the minimal polynomial $p_n(x)$ of ζ_n are exactly the primitive nth roots of 1. Show that the field extension $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ is Galois and prove its Galois group is $(\mathbb{Z}/n\mathbb{Z})^*$. How many subfields are there of $\mathbb{Q}(\zeta_{20})$?

(8) Let $\{e_1, \ldots, e_n\}$ be a basis of a real vector space V and let $\Lambda := \left\{\sum r_i e_i \,\middle|\, r_i \in \mathbb{Z}\right\}$. Let $\cdot$ be a non-degenerate ($v \cdot w = 0$ for all $w \in V \implies v = 0$) symmetric bilinear form on V such that the *Gram matrix* $M = (e_i \cdot e_j)$ has integer entries. Define the *dual* of Λ to be

$$\Lambda^\vee := \{v \in V \,|\, v \cdot x \in \mathbb{Z} \text{ for all } x \in \Lambda\}.$$

(a) Show that $\Lambda \subset \Lambda^\vee$.
(b) Prove that $\det M \neq 0$ and that the rows of M^{-1} span $\Lambda^\vee$.
(c) Prove that $\det M = |\Lambda^\vee/\Lambda|$.