By providing my signature below I acknowledge that this is my work, and I did not get any help from anyone else:

Name (sign): Solutions ____________ Name (print): ____________

Student Number: ____________

Instructor's Name: ____________ Meeting Time: ____________

Problem Number	Points Possible	Points Made
1	60	
2	30	
3	10	
4	10	
5	20	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	180	

- If you need extra space use the last page.
- Please show your work. **An unjustified answer may receive little or no credit.**
- Your test must be **neat**. We will take off points for sloppiness.
- If it is determined that you copied the work of another student you will receive no points for this test.
- The total number of possible points that is assigned for each problem is shown here. The number of points for each subproblem is shown within the exam.
- Please turn off your mobile phone.
- You are not allowed to use a calculator with graphing capabilities or the ability to manipulate expressions.

1. Determine the derivatives of each of the following functions:

   (a) [10 pts] $f(x) = x^2 \sin(4x) + 4$

$$f'(x) = 4x^2\cos(4x) + 2x\sin(4x)$$

   (b) [10 pts] $g(x) = \dfrac{e^{3x}+1}{1+5x} + 2$ $= (e^{3x}+1)(1+5x)^{-1} + 2$

$$g'(x) = \frac{(1+5x)(3e^{3x}) - (e^{3x}+1)(5)}{(1+5x)^2} = (e^{3x}+1)\left(-5(1+5x)^{-2}\right) + (1+5x)^{-1}(3e^{3x})$$

   (c) [10 pts] $h(x) = x^3 \ln\left(4 + x\cos(2x+1)\right)$

$$h'(x) = x^3 \cdot \frac{1}{4+x\cos(2x+1)} \cdot \left(-2x\sin(2x+1) + \cos(2x+1)\right) + 3x^2 \ln\left(4 + x\cos(2x+1)\right)$$

$$h'(x) = \frac{x^3\left(-2x\sin(2x+1) + \cos(2x+1)\right)}{4 + x\cos(2x+1)} + 3x^2 \ln\left(4 + x\cos(2x+1)\right)$$

(d) [10 pts] $p(t) = \frac{1}{t} \cdot \frac{1+\sqrt{t}}{5} \cdot (1+t)^4 = \frac{1}{5}\left(\frac{1+t^{1/2}}{t}\right)(1+t)^4 = \frac{1}{5}(t^{-1}+t^{-1/2})(1+t)^4$

$p'(t) = \frac{1}{5}\left[(t^{-1}+t^{-1/2})\cdot 4(t+1)^3 + (1+t)^4\left(-t^{-2} - \frac{1}{2}t^{-3/2}\right)\right]$

(e) [10 pts] $s(t) = \frac{\sin(3t)\tan(4t)}{t}$

$s'(t) = \frac{t(\sin(3t)\cdot 4\sec^2(4t) + 3\tan(4t)\cos(3t)) - \sin(3t)\tan(4t)}{t^2}$

(f) [10 pts] $q(t) = t^2 \ln(t^2+1)$

$q'(t) = t^2 \cdot \frac{1}{t^2+1} \cdot 2t + 2t\ln(t^2+1)$

2. Determine the anti-derivative represented by each of the following indefinite integrals.

(a) [10 pts] $\int \left(e^{x/2} + \sin(5x)\right)\, dx$

$= 2e^{x/2} - \frac{1}{5}\cos(5x) + C$

(b) [10 pts] $\int \left(2x - \frac{4}{x} + 1\right)\, dx$

$= x^2 - 4\ln|x| + x + C$

(c) [10 pts] $\int \frac{\ln(x)}{x}\, dx = \int \ln(x) \cdot \frac{1}{x}\, dx = \int u\, du = \frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{2}(\ln(x))^2 + C}$

$u = \ln(x)$
$du = \frac{1}{x}\, dx$

3. A timer will be constructed using a pendulum. The period in seconds, T, for a pendulum of length L meters is

$$T = 2\pi\sqrt{L/g}, = 2\pi\sqrt{L/9.81}$$

where g is 9.81 m/sec. The error in the measurement of the period, $\triangle T$, should be ± 0.05 seconds when the length is 0.2 m.

(a) [5 pts] Determine the exact resulting error, $\triangle L$, necessary in the measurement of the length to obtain the indicated error in the period.

$\Delta T = \pm .05$ when $L = 0.2$

$$\frac{T}{2\pi} = \sqrt{\frac{L}{9.81}}$$

$$T = 2\pi\sqrt{\frac{L}{9.81}}$$

$$\Delta T = 2\pi\sqrt{\frac{L_2}{9.81}} - 2\pi\sqrt{\frac{L_1}{9.81}}$$

$L_2 = 0.2 + \Delta L$, $L_1 = 0.2$

$\Delta T = \pm .05$

$$\pm .05 = 2\pi\sqrt{\frac{0.2+\Delta L}{9.81}} - 2\pi\sqrt{\frac{0.2}{9.81}}$$

$$2\pi\sqrt{\frac{0.2}{9.81}} \pm .05 = 2\pi\sqrt{\frac{0.2+\Delta L}{9.81}}$$

$$\left(\sqrt{\frac{0.2}{9.81}} \pm \frac{.05}{2\pi}\right)^2 = \left(\sqrt{\frac{0.2+\Delta L}{9.81}}\right)^2$$

$$9.81\left(\sqrt{\frac{0.2}{9.81}} \pm \frac{.05}{2\pi}\right)^2 - 0.2 = \Delta L$$

(b) [5 pts] Use the linearization of the period in the formula above to **estimate** the error, $\triangle L$, necessary in the measurement of the length to obtain the indicated error in the period.

$\Delta L = dL$, $\Delta T \approx dT$

$$T = \frac{2\pi}{\sqrt{9.81}} \cdot \sqrt{L}$$

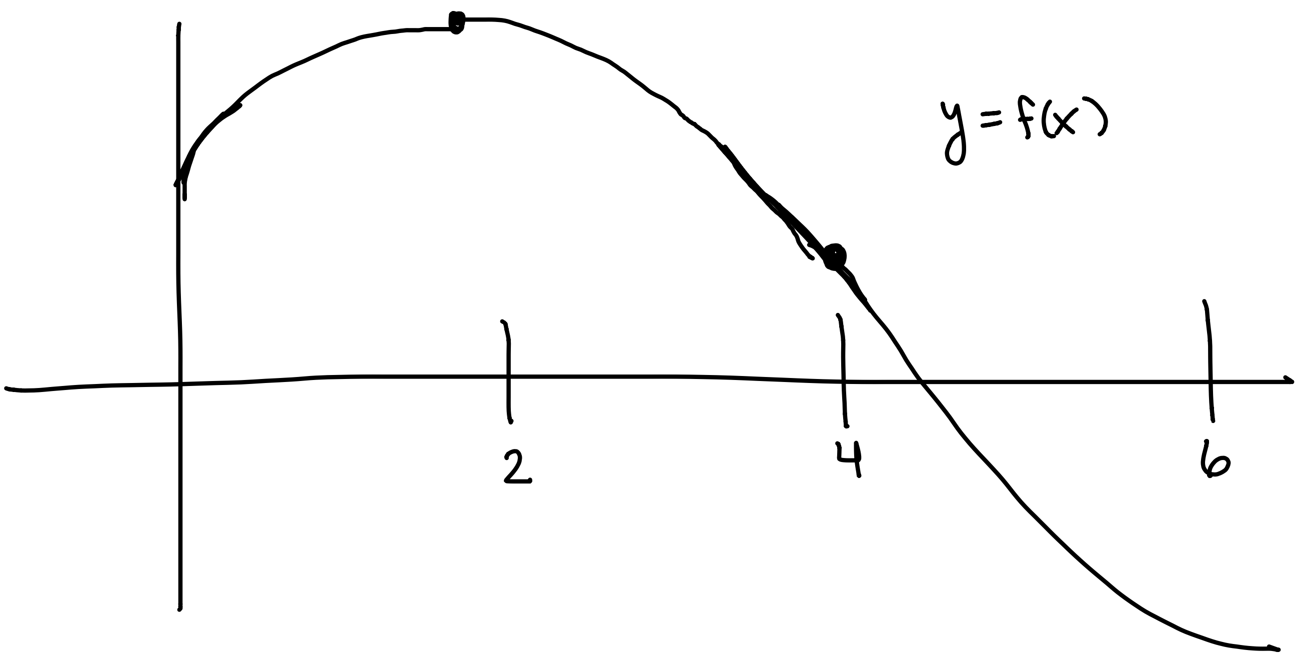
$$dT = \frac{\cancel{2}\pi}{\sqrt{9.81}} \cdot \frac{1}{\cancel{2}\sqrt{L}}\, dL$$

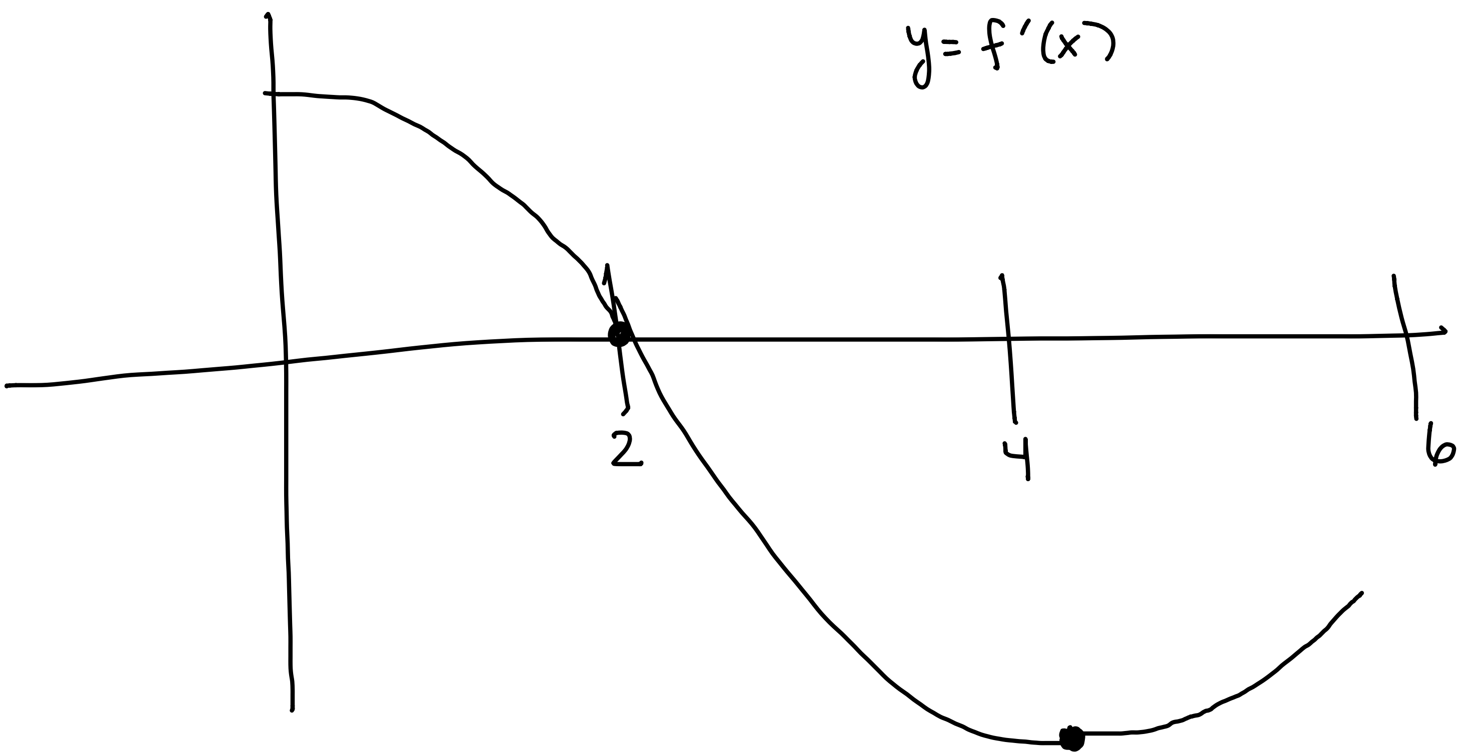
$$\pm .05 \approx \frac{\pi}{\sqrt{9.81}\sqrt{0.2}}\, dL$$

$$dL \approx \pm \frac{.05(\sqrt{9.81})(\sqrt{0.2})}{\pi}$$

approximately ΔL

4. [10 pts] A function, $f(x)$, is concave down and increases for $0 < x < 2$. It is concave down and decreases for $2 < x < 4$. It is concave up and decreases for $4 < x < 6$. Make a sketch of a function that satisfies this criteria. On a separate set of axes below it make a sketch of the function's derivative.





 Points deducted: ____
out of a possible 10 points

5. Determine each of the following limits.

(a) [10 pts] $\lim_{x\to 0}(1+x)^{3/x} = \boxed{e^3}$

$\lim_{x\to 0} \ln\left[(1+x)^{3/x}\right] = \lim_{x\to 0} \frac{3}{x}\ln(1+x) = \lim_{x\to 0} \frac{3\ln(1+x)}{x}$

IF $\frac{0}{0}$

$\overset{LH}{=} \lim_{x\to 0} \frac{3\cdot\frac{1}{1+x}}{1} = 3$ so answer to original is e^3

(b) [10 pts] $\lim_{x\to\infty} \frac{\cos(2x)-1}{x^2}$.

$-1 \le \cos(2x) \le 1$

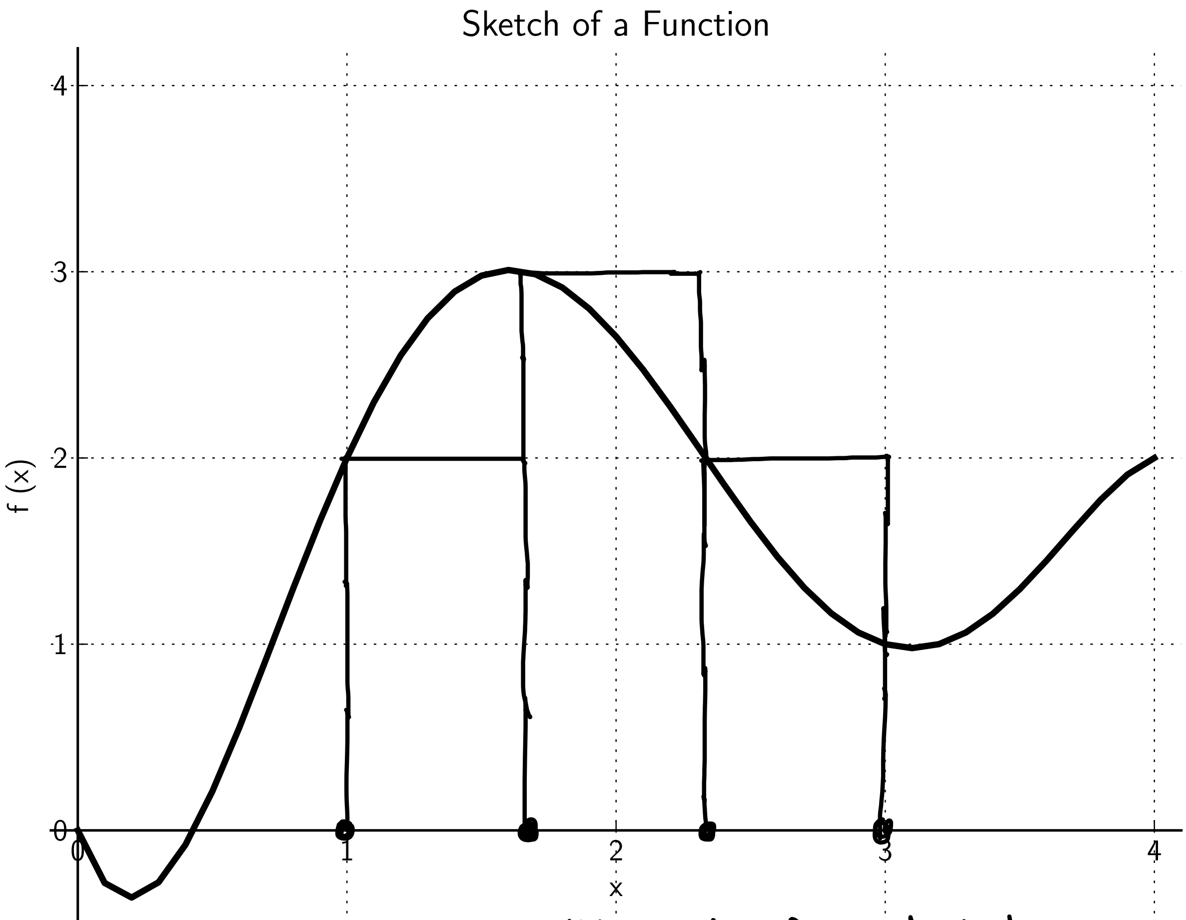
$-2 \le \cos(2x) - 1 \le 0$

$x \to \infty$ means we can assume $x > 0$, so

$\frac{-2}{x^2} \le \frac{\cos(2x)-1}{x^2} \le 0$, since $\frac{0}{x^2} = 0$

Since $\lim_{x\to\infty} \frac{-2}{x^2} = 0$ and $\lim_{x\to\infty} 0 = 0$, according to the Squeeze (sandwich) Theorem, $\lim_{x\to\infty} \frac{\cos(2x)-1}{x^2} = 0$.

6. [10 pts] The graph of a function is shown below. Approximate the area under the curve from $x = 1$ to $x = 3$ using a Riemann sum with three intervals. Add a sketch of the rectangles to the plot that correspond to the Riemann sum. Show all of your work. (You do not have to evaluate your result and can leave it as a sum, but it must be in a form that can be directly entered into a calculator.)



Using 3 subintervals of equal width and left endpoints:

$\Delta x = \frac{3-1}{3} = \frac{2}{3}$ $[1, 5/3]$ $[5/3, 7/3]$ $[7/3, 3]$

Riemann Sum: $f(1)\cdot\frac{2}{3} + f\left(\frac{5}{3}\right)\cdot\frac{2}{3} + f\left(\frac{7}{3}\right)\cdot\frac{2}{3}$

$= 2\cdot\frac{2}{3} + 3\cdot\frac{2}{3} + 2\cdot\frac{2}{3} = \boxed{\frac{14}{3}}$

(There are many other correct Riemann sums, for example using right endpoints instead.)

7. [10 pts] Use the definition of the derivative to show that

$$\frac{d}{dx}\left(x + \frac{1}{x} + 2\right) \;=\; 1 - \frac{1}{x^2}.$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\left(x+h+\frac{1}{x+h}+2\right)-\left(x+\frac{1}{x}+2\right)}{h}$$

$$= \lim_{h\to 0} \frac{h+\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0}\left(1+\frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right)\right)$$

$$= \lim_{h\to 0}\left(1+\frac{1}{h}\left(\frac{x}{(x+h)(x)}-\frac{x+h}{(x+h)(x)}\right)\right)$$

$$= \lim_{h\to 0}\left(1+\frac{1}{h}\left(\frac{-h}{(x+h)(x)}\right)\right)$$

$$= \lim_{h\to 0}\left(1-\frac{1}{(x+h)(x)}\right)$$

$$= 1 - \frac{1}{x^2}$$

8. [10 pts] The velocity of an object is given by

$$v(t) \quad = \quad \sin(2t) + e^{-t}.$$

The initial position is $x(0) = 2\text{m}$. Determine the equation for the position at any time.

$$s(t) = \frac{-1}{2}\cos(2t) - e^{-t} + C$$

$$2 = s(0) = \frac{-1}{2}\cos(0) - e^{0} + C$$

$$2 = \frac{-1}{2} - 1 + C$$

$$2 = \frac{-3}{2} + C$$

$$\frac{7}{2} = C$$

$$s(t) = \frac{-1}{2}\cos(2t) - e^{-t} + \frac{7}{2}$$

9. A container holds 50 liters of water. A valve is closed and is slowly turned. The water is drained from the container in 2 minutes.

   For each statement indicate if it must be true, must be false, or if it is not possible to determine indicate that you cannot tell from the given information. For each statement provide a complete, one sentence explanation for your reasoning.

   (a) [3 pts] **True/False/Cannot Tell** The container held 25 liters of water after one minute.

   Cannot Tell: At some point the container held 25 liters, but it may or may not have been at one minute.

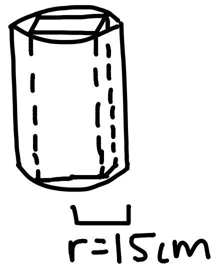
   (b) [4 pts] **True/False/Cannot Tell** At some point in time the rate of change of the volume of water in the container was 25 l/min.

   False: The volume is decreasing so the rate of change can't be positive. However, the average rate of change of the volume is $\frac{V(2)-V(0)}{2-0} = \frac{0-50}{2} = -25$ L/min, so the mean value theorem guarantees a rate of change of -25 L/min at some time.

   (c) [3 pts] **True/False/Cannot Tell** At some point in time the rate of change of the volume of water in the container was 20 l/min.

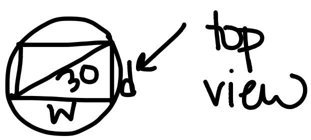
   False: The rate of change of the volume can't be positive. However, since $V'(0)=0$ and $V'(c)=-25$ for some c in $(0,2)$ by part (b), the Intermediate Value Theorem applied to V' means that at some time the rate of change of V is -20 L/min.

10. [10 pts] A rectangular beam will be cut from a cylindrical log whose radius is 15cm. The stiffness of the resulting beam is proportional to the width multiplied by the cube of its depth. Determine the width and depth that will result in the beam with the greatest stiffness.



S = stiffness

$S = k\,wd^3$, where k is a constant



$w^2 + d^2 = (30)^2$

$d = \sqrt{900 - w^2}$

$S = k\,w\left(\sqrt{900-w^2}\right)^3 = kw\,(900-w^2)^{3/2}$

domain: $[0, 30]$

$S' = k\cdot\left(w\cdot\frac{3}{2}(900-w^2)^{1/2}\cdot -2w + (900-w^2)^{3/2}\right)$

$S' = k\sqrt{900-w^2}\left(-3w^2 + 900 - w^2\right) = k\sqrt{900-w^2}\left(900 - 4w^2\right)$

Critical number(s):
S' dne: none $S' = 0$: $0 = 900 - 4w^2$
$4w^2 = 900$
$w^2 = \frac{900}{4} = \left(\frac{30}{2}\right)^2 = 15^2$
$w = \cancel{-15}, 15$

w	S
0	0
15	$k\cdot 15\left(\sqrt{900-225}\right)^3$
30	0

max stiffness, so use $w = 15$ cm and $d = \sqrt{900-(15)^2}$ cm

Extra space for work. If you want us to consider the work on this page you should write your name, instructor and meeting time below.

Name (print): ____________ Instructor: Name (print): ____________ Time: ____________